5CCS2FC2: Foundations of Computing II

The Limits of Computation

Week 10

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The Entscheidungsproblem

(The Decision Problem)

The Entscheidungsproblem

The Entscheidungsproblem

Input) Given a *Predicate* formula F

Output) **True** if and only if F is a tautology.

The Entscheidungsproblem

> **Theorem** The Entscheidungsproblem is undecidable.

Proof:

Step 1) In the proof of the **Cook-Levin Theorem**, for a given machine M and input word w, we constructed a **propositional formula** $F_{M,w}$ such that

$F_{M,w}$ is satisfiable	$\iff$	M accepts w within a *polynomial* number of steps

The Entscheidungsproblem

Step 2) Unfortunately, we could not express **global** properties such as

"*The machine M will eventually accept*" without **infinitely long** 'disjunctions'!

$$\boxed{S_{q,0} \lor S_{q,1} \lor S_{q,2} \lor S_{q,3} \lor S_{q,4} \lor S_{q,5} \lor S_{q,6} \lor \ldots}$$

(where $q = q_{\text{accept}}$ is the accepting state)

Step 3) However, with a suitable choice of **predicate symbols**

$$
\begin{aligned}
C_a(i,t) \quad &= \quad \text{Cell } i \text{ of the } M\text{'s tape contains symbol } a \text{ at time } t \\
H(i,t) \quad &= \quad \text{The tape head is in position } i \text{ at time } t \\
S_q(t) \quad &= \quad \text{The machine is in state } q \text{ at time } t
\end{aligned}
$$

we could write this down this **infinite 'disjunction'** as

$$
S_q(0) \;\vee\; S_q(1) \;\vee\; S_q(2) \;\vee\; S_q(3) \;\vee\; S_q(4) \;\vee\; \ldots \qquad \mapsto \qquad \exists t\; S_q(t)
$$

Step 4) Hence we can write down a formula $F_{M,w}$ in **predicate logic** such that

$F_{M,w}$ is satisfiable $\iff$ M *eventually* accepts input w

Q.E.D.

This is effectively a polynomial reduction between the **Accepting problem** at the complement of the **Entscheidungsproblem**. Since the former is undecidable, so must be the latter.

End of Slides!

The Entscheidungsproblem

230 A. M. Turing [Nov. 12,

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions

End of Slides!