5CCS2FC2: Foundations of Computing II

P vs NP: The Million Dollar Problem

Week 2

Dr Christopher Hampson

Department of Informatics

King's College London

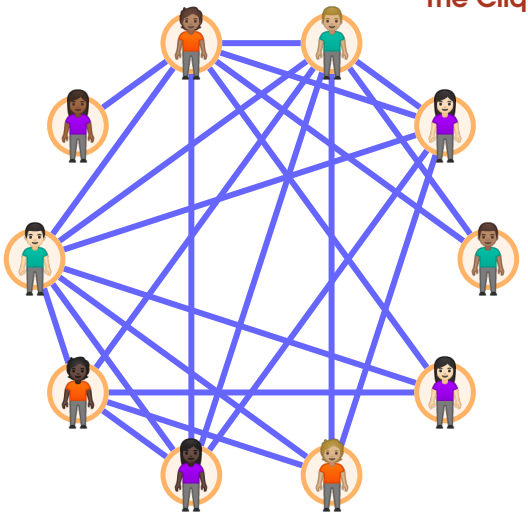
The Clique Problem

The Clique Problem CLIQUE

Input) — An undirected graph $G = (V, E)$,
— an integer $k > 2$,

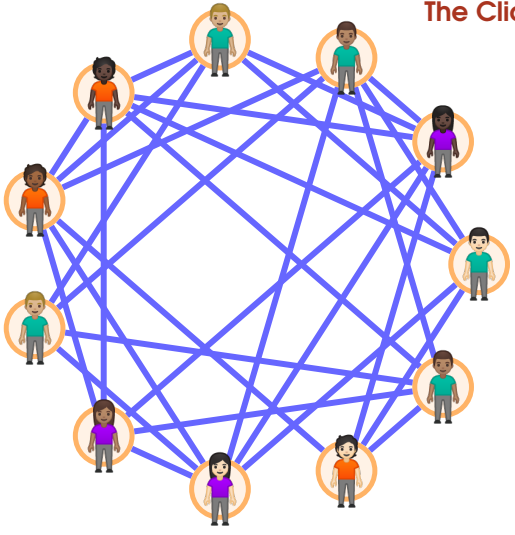
Output) **True** if and only if G contains a *clique* of size k

The Clique Problem



$$k = 5$$

The Clique Problem



$$k = 4$$

The Clique Problem

Theorem The Clique Problem **CLIQUE** belongs to **NP**.

Proof: **Input)** Given an undirected graph $G = (V, E)$ and integer $k > 2$,

Step 1) We can decide whether G contains a clique of size k by checking **every subset** of vertices of size k.

However there are $\binom{n}{k}$ possible subsets — **NOT** polynomial!

The Clique Problem

Theorem The Clique Problem **CLIQUE** belongs to **NP**.

Step 2) However, a non-deterministic machine can check each subset in **parallel**,

Each subset requires us to check only k^2 edges — **polynomial!**

Q.E.D.

The Clique Problem

Theorem The Boolean Satisfiability problem is polynomially reducible to the Clique finding problem. *i.e.* **SAT** $\leq_p$ **CLIQUE**

Proof: **Input)** Given formula F in Conjunctive Normal Form,

Step 1) We want to **construct** a graph $G_F = (V, E)$ and **choose** an integer $k > 2$ such that

F is satisfiable $\iff$ G_F has a clique of size k.

Step 2) Define $G_F = (V, E)$ as follows

$$F = (P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R)$$

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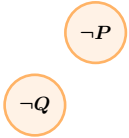

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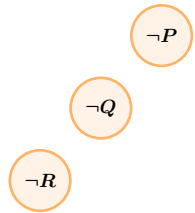

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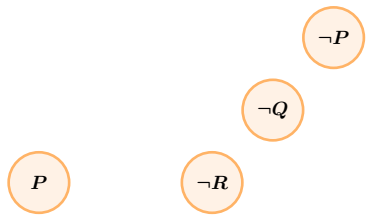

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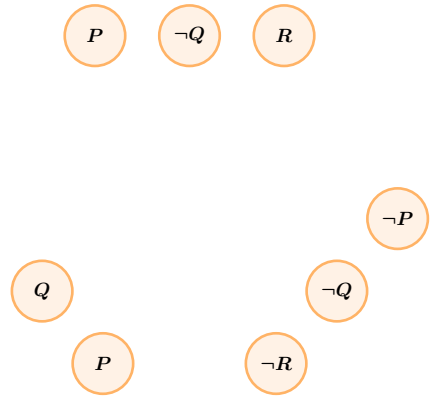

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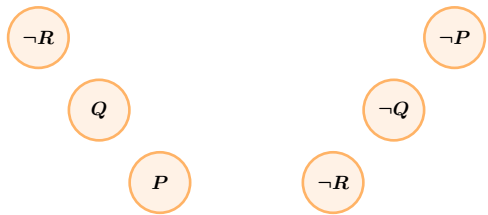

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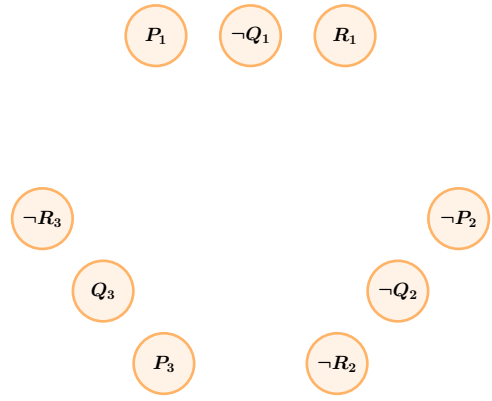

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The Clique Problem

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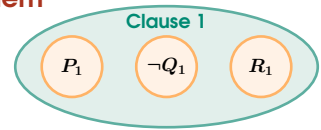
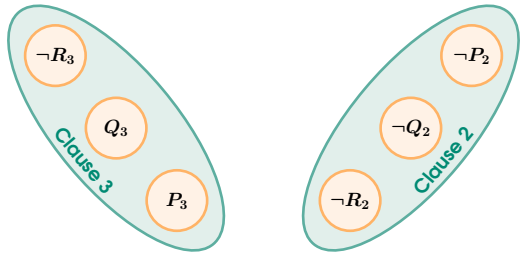




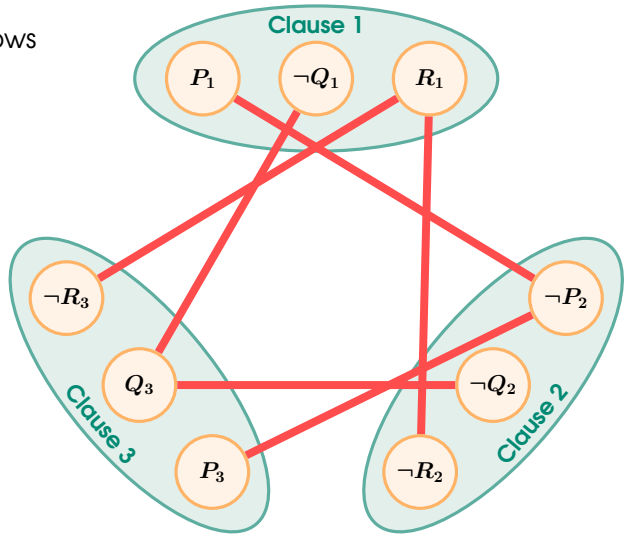
The Clique Problem

Step 2) Define $G_F = (V, E)$ as follows

$$\left(L_1^i,\, L_2^j\right) \in E$$

$$\Longleftrightarrow$$

$$i \neq j \quad \text{and} \quad L_1 \not\equiv \neg L_2$$



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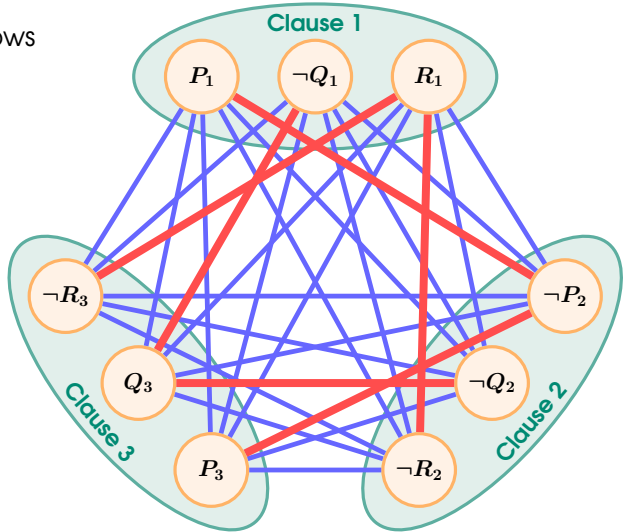

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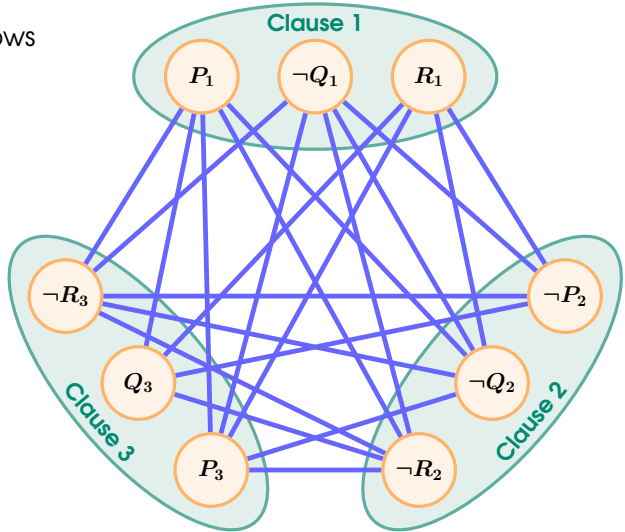
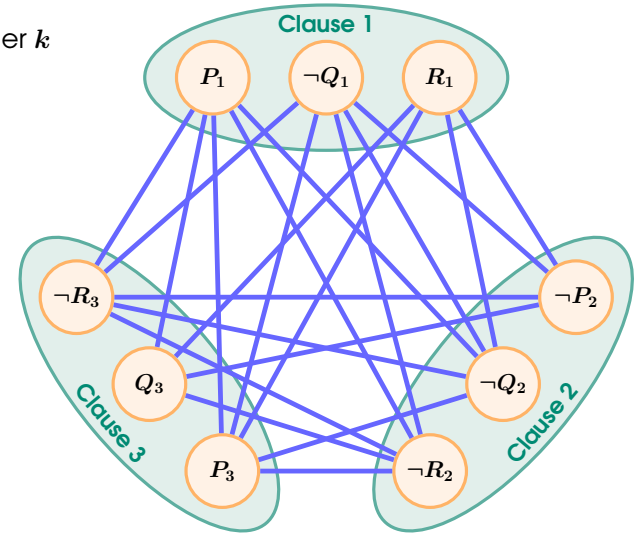


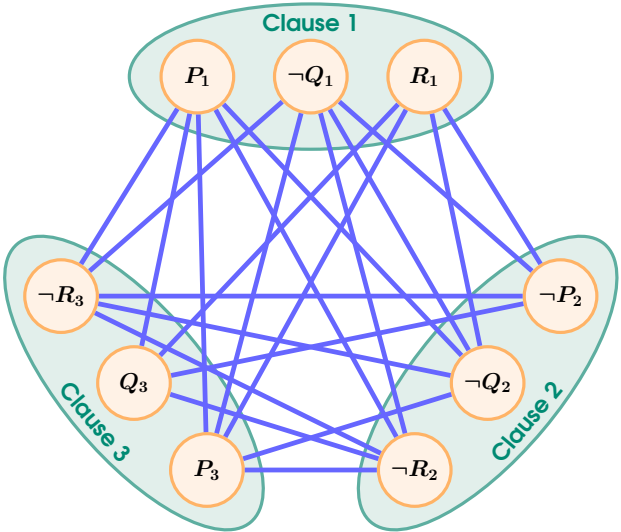
The Clique Problem

Step 3) Choose appropriate integer k

$$k = \#\text{ clauses in } F$$



The Clique Problem



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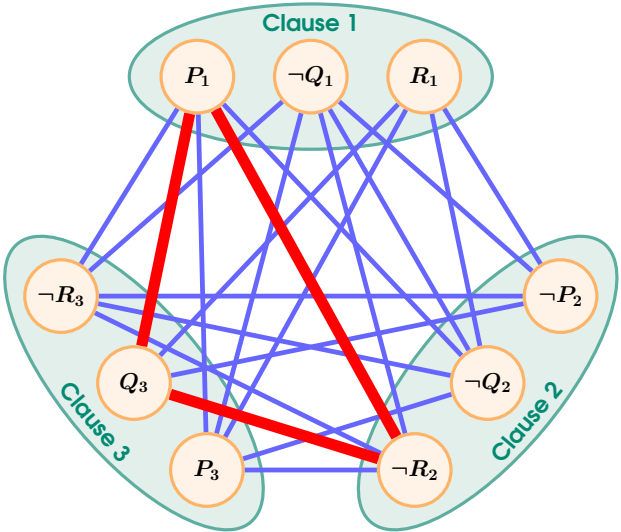

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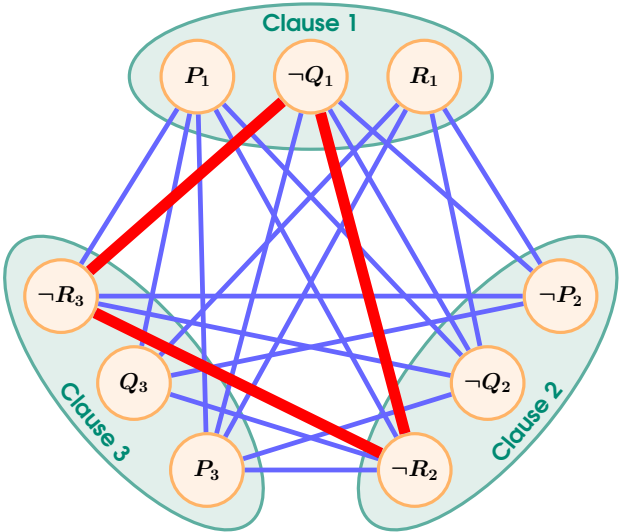

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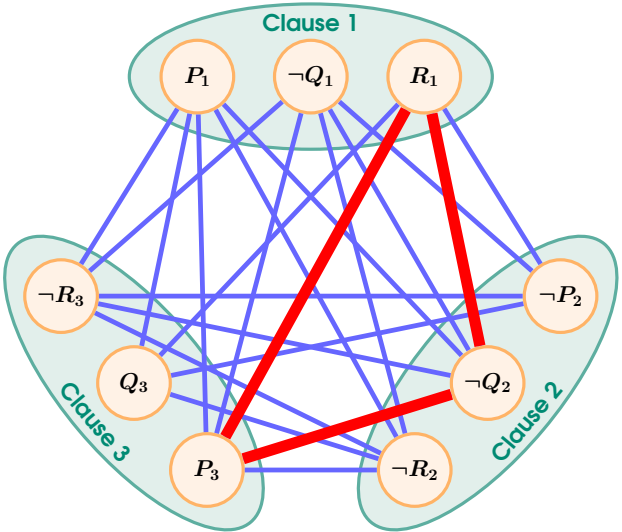

Step 4) Hence, we have that

F is satisfiable $\iff$ G_F contains a clique of size k

SAT $\leq_p$ **CLIQUE**

Q.E.D.

The Clique Problem

Theorem **CLIQUE** is **NP**-complete.

End of Slides!