5CCS2FC2: Foundations of Computing II

P vs NP: The Million Dollar Problem

Week 3

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The Cook-Levin Theorem

(SAT is NP-complete)

The Cook-Levin Theorem

Theorem (The Cook-Levin Theorem) The Boolean Satisfiability problem **SAT** is **NP**-complete.

Proof: We want to show that **every** problem $X \in$ **NP** can be reduced to **SAT**.

Step 1) Let $X \in$ **NP** be **any problem** belonging to the class **NP**.

By **definition** there is some NDTM M such that

M has a polynomial-length computation that accepts w	$\iff$	$w \in X$

The Cook-Levin Theorem

Step 2) For each **state** $q \in Q$, **symbol** $a \in \Sigma$ and **integers** $i, t \leq T_M(n)$

$$C_{i,t,a} = \text{Cell } i \text{ of the } M\text{'s tape contains symbol } a \text{ at time } t$$

$$H_{i,t} = \text{The tape head is in position } i \text{ at time } t$$

$$S_{q,t} = \text{The machine is in state } q \text{ at time } t$$

The Cook-Levin Theorem

- *"The machine cannot be both in state q_4 and state q_5 at time $t = 2$"*

$$\boxed{\neg(S_{q_4,2} \wedge S_{q_5,2})}$$

- *"At time $t = 3$ the tape head must be in positions 0 or 1 or 2 or 3"*

$$\boxed{\left(H_{0,3} \vee H_{1,3} \vee H_{2,3} \vee H_{3,3}\right)}$$

The Cook-Levin Theorem

- *"At time $t = 3$ if cell 1 contains an a then it must not contain a b"*

$$\boxed{C_{3,1,a} \rightarrow \neg C_{3,1,b}}$$

- *"At time $t = 6$ if the machine is in state q_1 and the tape head is in position 4 and cell 4 contains an a at time $t = 6$ then at time $t = 7$ the machine will be in state q_2, the tape head will move to position 3 and cell 4 will contain a b."*

$$\boxed{(S_{q_1,6} \wedge H_{4,6} \wedge C_{4,6,a}) \rightarrow (S_{q_2,7} \wedge H_{3,7} \wedge C_{4,7,b})}$$

The Cook-Levin Theorem

Step 3) With these **propositional variables**, we can construct a formula:

$$F_{M,w} \quad = \quad \text{M has a polynomial-length computation that accepts w}$$

$$\text{M has a polynomial-length computation that accepts w} \iff w \in X$$

The Cook-Levin Theorem

Step 4) We then have that

$$w \in X \iff F_{M,w} \text{ is satisfiable}$$

$$X \leq_p \textbf{SAT}$$

$$X \leq_p \textbf{SAT} \quad \text{for all } X \in \textbf{NP}$$

Q.E.D.

The Cook-Levin Theorem

Theorem **SAT** is **NP**-complete.

End of Slides!