5CCS2FC2: Foundations of Computing II

P vs NP: The Million Dollar Problem

Week 2

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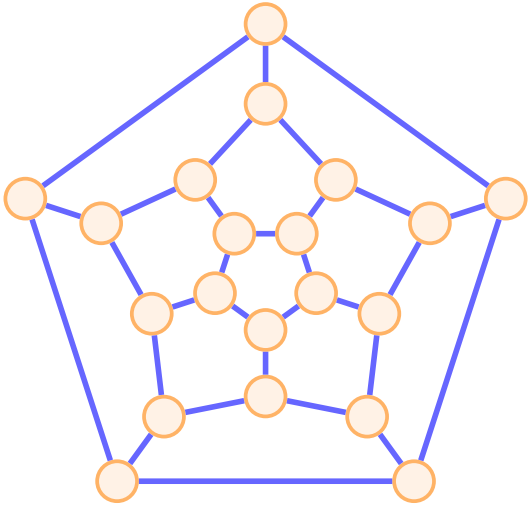
Hamiltonian Cycle Problem

The Hamiltonian Cycle Problem HAMILTONIAN

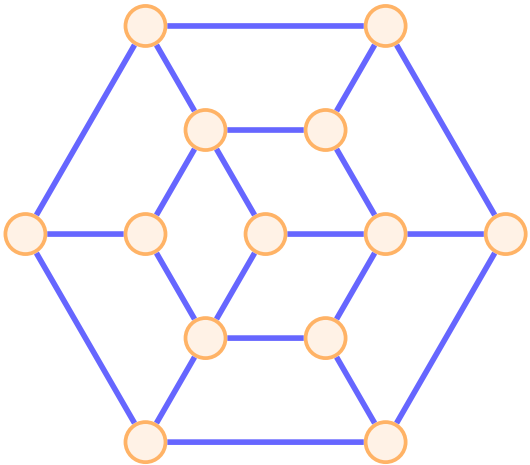
Input) An undirected graph $G = (V, E)$,

Output) **True** if and only if G contains a *Hamiltonian Cycle*

Hamiltonian Cycle Problem



Hamiltonian Cycle Problem



Hamiltonian Cycle Problem

Theorem The Hamiltonian Cycle Problem **HAMILTONIAN** belongs to **NP**.

Proof: **Input)** Given an undirected graph $G = (V, E)$,

Step 1) We can decide whether G contains a Hamiltonian cycle by checking **every permutation** of the vertices.

However there are $n!$ possible permutations — **NOT** polynomial!

Hamiltonian Cycle Problem

Theorem The Hamiltonian Cycle Problem **HAMILTONIAN** belongs to **NP**.

Step 2) However, a NDTM can check each permutation in **parallel**,

Each permutations requires us to check only n edges — **polynomial!**

Q.E.D.

Hamiltonian Cycle Problem

Theorem The Boolean Satisfiability problem is polynomially reducible to the Hamiltonian cycle problem.

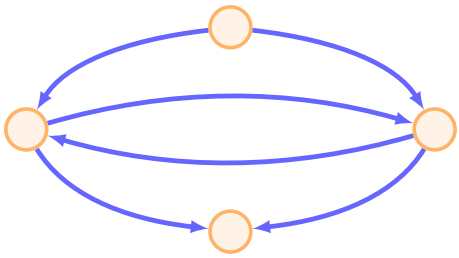
Proof: **Input)** Given a formula F in Conjunctive Normal Form

Step 1) We want to **construct** a graph G_F such that

F is satisfiable $\iff$ G_F has a Hamiltonian cycle

Hamiltonian Cycle Problem

Step 2) We will make use of the following **gadget**



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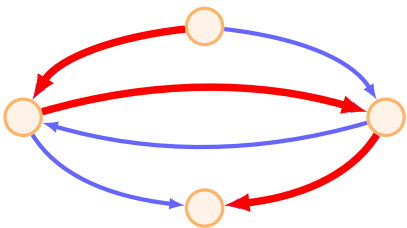
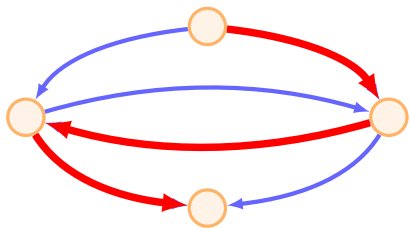
Step 3) Note that there are only two possible **Hamiltonian paths** through the gadget



OR



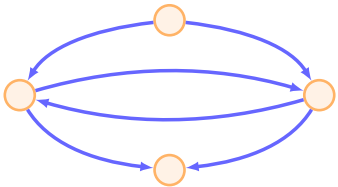
Hamiltonian Cycle Problem

Step 4) We then proceed as follows...

Hamiltonian Cycle Problem

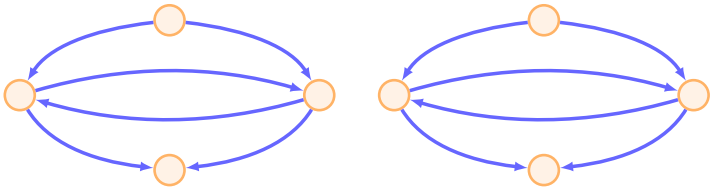
$$F = (P \vee \neg Q) \wedge (\neg P \vee \neg R)$$

Hamiltonian Cycle Problem



$$F \;=\; (P \vee \neg Q) \;\wedge\; (\neg P \vee \neg R)$$

Hamiltonian Cycle Problem



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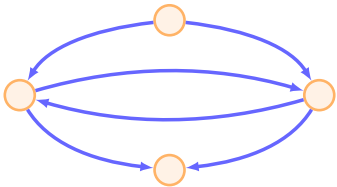
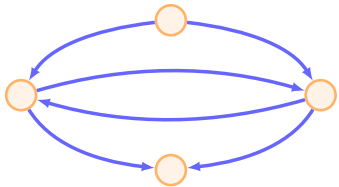
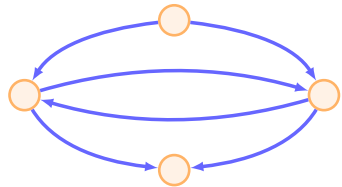
Hamiltonian Cycle Problem

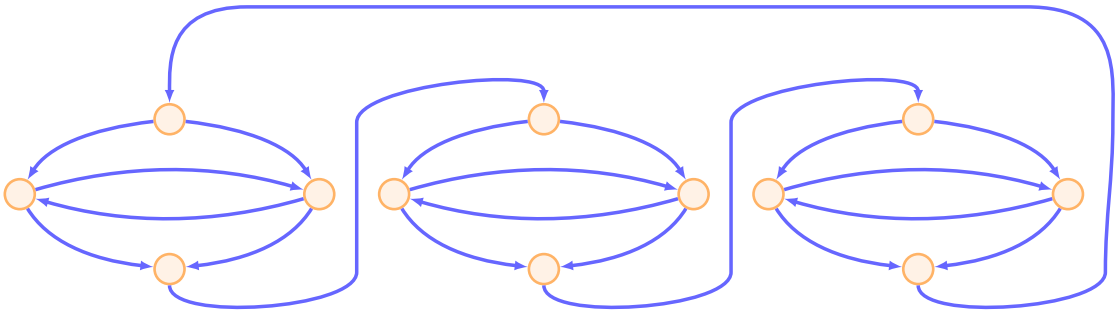






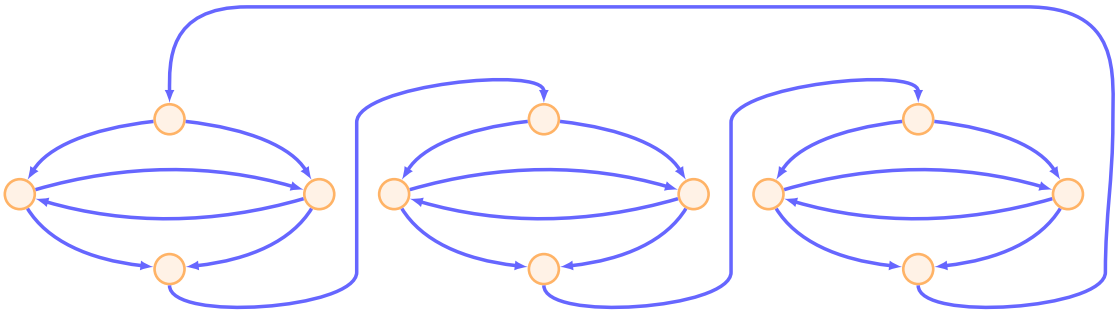
$$F = (P \vee \neg Q) \wedge (\neg P \vee \neg R)$$

Hamiltonian Cycle Problem



$$F = (P \vee \neg Q) \wedge (\neg P \vee \neg R)$$

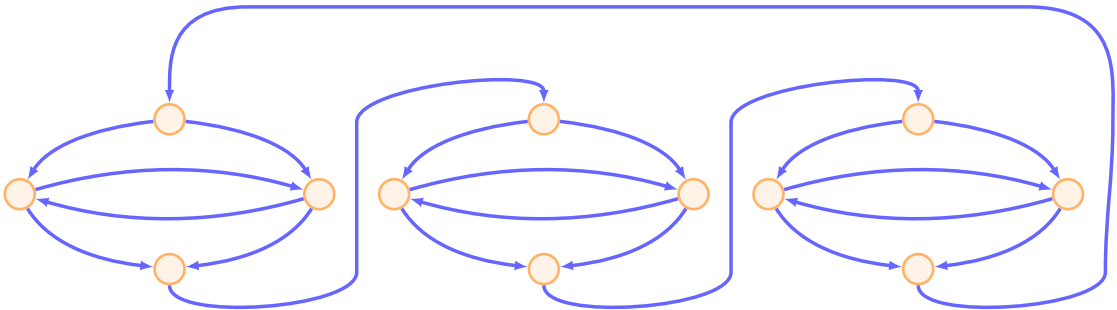
Hamiltonian Cycle Problem



$P \vee \neg Q$

$F = \qquad\qquad \wedge\ (\neg P \vee \neg R)$

Hamiltonian Cycle Problem

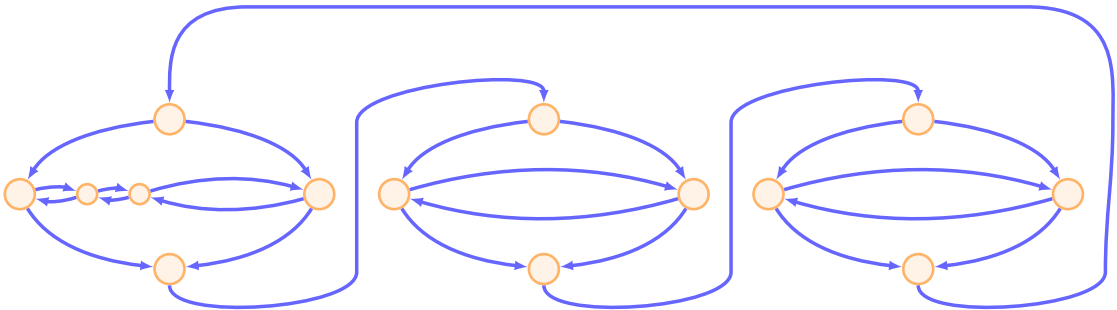


$P \vee \neg Q$

$\neg P \vee \neg R$

$F = \qquad \wedge$

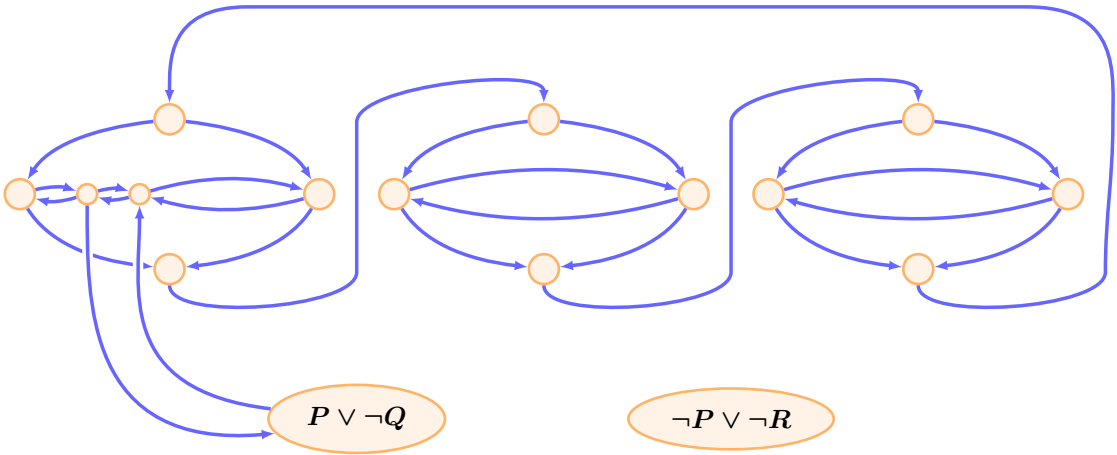
Hamiltonian Cycle Problem



$P \vee \neg Q$

$\neg P \vee \neg R$

Hamiltonian Cycle Problem



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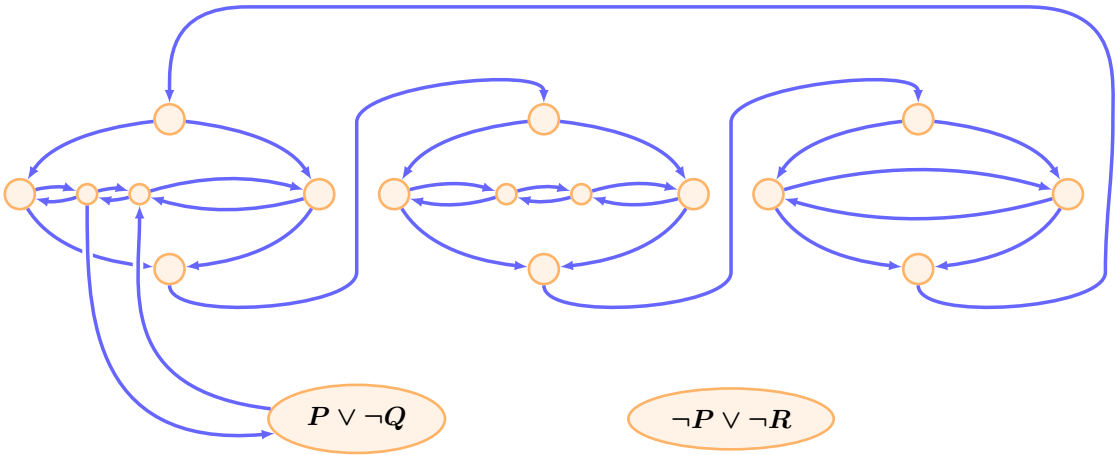

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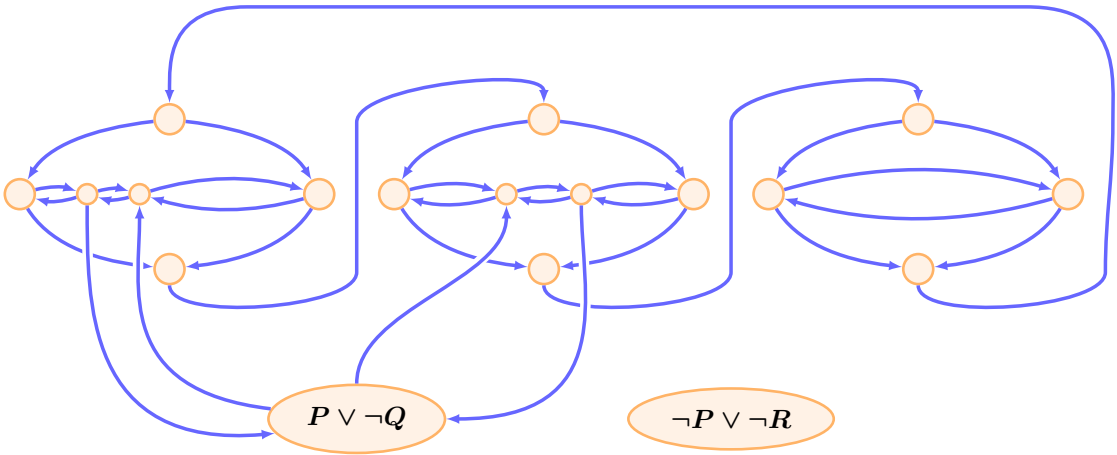

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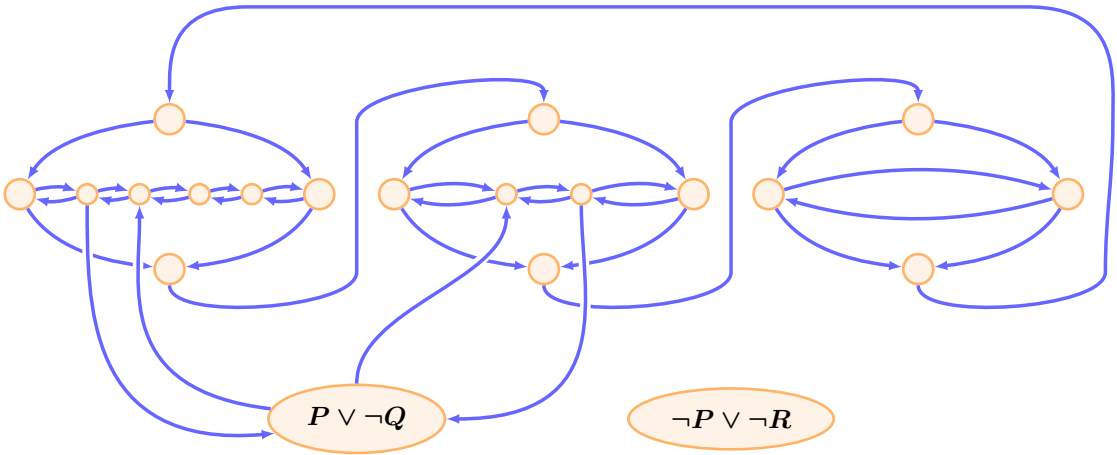

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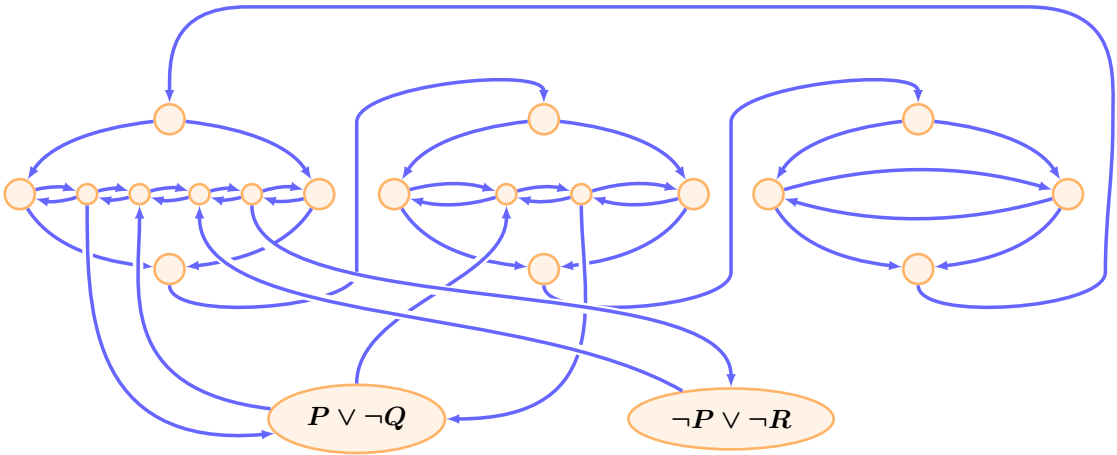

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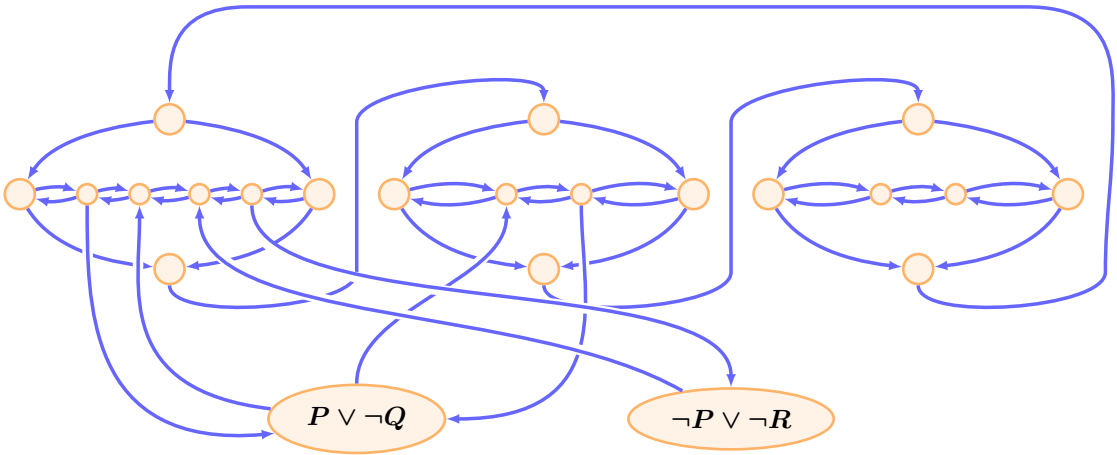

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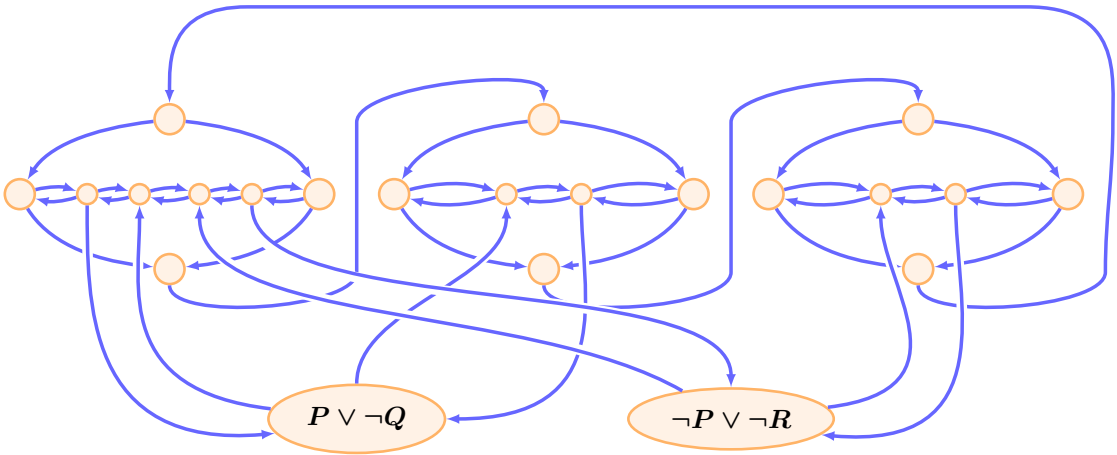

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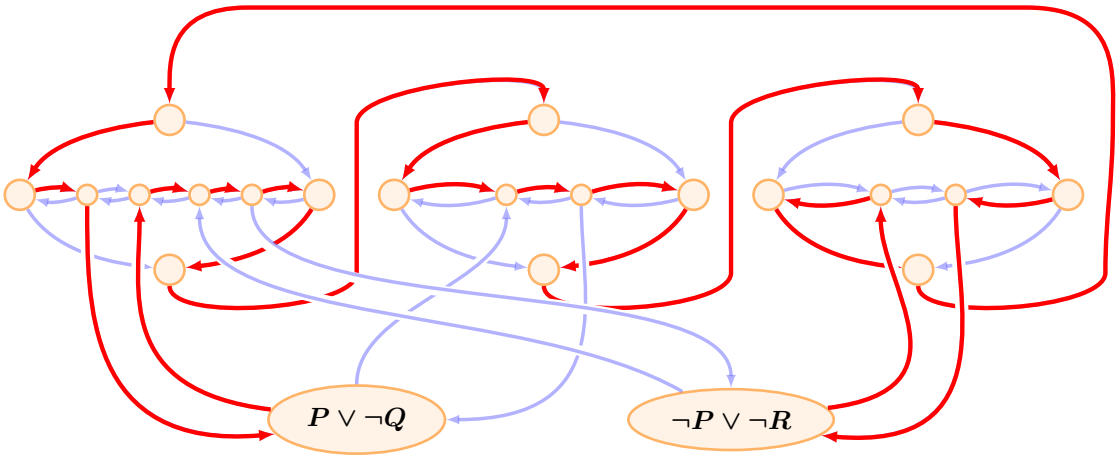

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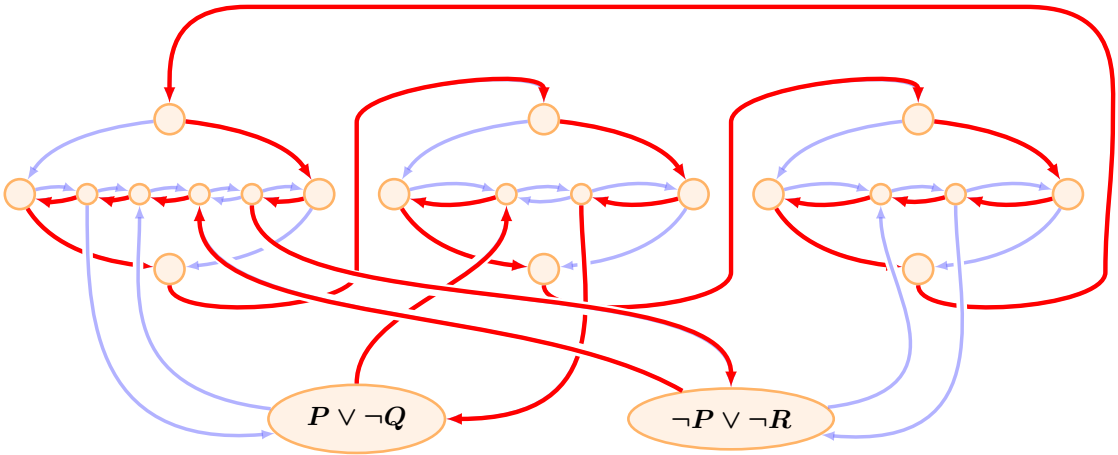

Step 5) Hence, we have that

F is satisfiable $\iff$ G_F contains a Hamiltonian cycle

SAT $\leq_p$ **HAMILTONIAN**

Q.E.D.

Hamiltonian Cycle Problem

Theorem **HAMILTONIAN** is **NP**-complete.

End of Slides!