5CCS2FC2: Foundations of Computing II

Divide-and-Conquer and Recursion

Week 3

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Method of Substitution
i.e. Proof by Induction

Proof by Induction

Proof by Induction

Base Case) Show that your solution holds for $n = 1$,

Inductive Case) **(i)** Assume your result holds for $n = k$,

(ii) Substitute to confirm that it also holds for $n = (k + 1)$.

Proof by Induction

Theorem Let $T(n)$ be a recursion relation defined by

$$T(1) = 1, \qquad T(n) = 2T(n-1) - 1$$

for all $n > 1$. Show that $T(n) = 2^n - 1$, for all $n \geq 1$

Proof by Induction

Proof:

Base Case) First show that $T(n) = 2^n - 1$ for $n = 1$.

Left-hand-side		Right-hand-side
$T(1) = 1$	$=$	$2^1 - 1 = 1$

Proof by Induction

Inductive Case) Assume that

$$\textbf{(I.H.)} \quad T(k) = 2^k - 1 \qquad \text{for some } k \geq 1$$

We then have that for $n = (k+1)$

$$T(n) = 2\,T\,(n-1) + 1$$

$$T(k+1) = 2\,T\,(k) + 1$$

Proof by Induction

$$
\begin{aligned}
T(k+1) &= 2\,T(k) + 1 \\
&= 2\left(2^k - 1\right) + 1 \\
&= \left(2 \cdot 2^k - 2\right) + 1 \\
&= \left(2^{k+1} - 2\right) + 1 \\
&= 2^{k+1} - 1
\end{aligned}
$$

Proof by Induction

Theorem Let $T(n)$ be a recursion relation defined by

$$T(1) = 2, \qquad T(n) = T(n-1) + n$$

for all $n > 1$. Show that $T(n) = \dfrac{n^2 + n}{2}$, for all $n \geq 1$

Proof by Induction

Proof:

Base Case) First show that $T(n) = \dfrac{n^2 + n}{2}$ for $n = 1$.

Left-hand-side

$$T(1) = 1$$

$$=$$

Right-hand-side

$$\frac{1^2 + 1}{2} = 1$$

Proof by Induction

Inductive Case) Assume that

$$\text{(I.H.)} \quad T(k) = \frac{k^2 + k}{2} \qquad \text{for some } k \geq 1$$

We then have that for $n = (k + 1)$

Proof by Induction

$$
\begin{aligned}
T(k+1) &= T(k) + (k+1) \\
&= \left(\frac{k^2+k}{2}\right) + (k+1) \\
&= \left(\frac{k^2+k}{2}\right) + \left(\frac{2(k+1)}{2}\right) \\
&= \left(\frac{k^2+k + 2k+2}{2}\right) \\
&= \left(\frac{k^2+3k+2}{2}\right) \\
&= \frac{(k+1)^2+(k+1)}{2}
\end{aligned}
$$

Q.E.D.

Strong Induction

Proof by Strong Induction

Proof by Strong Induction

Base Case) Show that your solution holds for $n = 1$,

Inductive Case) **(i)** Assume your result holds **for all** $m \leq k$,

(ii) Substitute to confirm that it also holds for $n = (k + 1)$.

Proof by Strong Induction

Theorem Let $T(n)$ be a recursion relation defined by

$$T(1) = 1, \qquad T(n) = 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + n$$

for all $n > 1$. Show that $T(n) \geq n \log_2 n$, for all $n \geq 1$

Proof by Strong Induction

Proof:

Base Case) First show that $T(n) \geq n \log_2 n$ for $n = 1$.

Left-hand-side		Right-hand-side
$T(1) = 1$	$\geq$	$1 \cdot \log_2(1) = 0$

Proof by Strong Induction

Inductive Case) Assume that

$$\textbf{(I.H.)} \quad T(m) \geq m \log_2 m \qquad \text{for all } m \leq k \text{ for some } k \geq 1$$

We then have that for $n = (k + 1)$

Proof by Strong Induction

$$
\begin{aligned}
T(k+1) &= 2\,T\left(\left\lceil \tfrac{k+1}{2} \right\rceil\right) + (k+1) \\
&\geq 2\left(\left\lceil \tfrac{k+1}{2} \right\rceil\right) \log_2\left(\left\lceil \tfrac{k+1}{2} \right\rceil\right) + (k+1) \\
&\geq 2\left(\tfrac{k+1}{2}\right) \log_2\left(\tfrac{k+1}{2}\right) + (k+1) \\
&= (k+1)\Big[\log_2(k+1) - 1\Big] + (k+1) \\
&= (k+1)\log_2(k+1)
\end{aligned}
$$

Q.E.D.

Proof by Strong Induction

Theorem Let $F(n)$ denote the nth *Fibonacci Number* given by

$$F(0) \; = \; 0, \qquad F(1) \; = \; 1, \qquad F(n) \; = \; F(n-1) + F(n-2)$$

for all $n > 1$. Show that

$$F(n) \; = \; \frac{\Phi^n - \phi^n}{\sqrt{5}}$$

for all $n \geq 0$, where $\Phi = \frac{1+\sqrt{5}}{2}$ and $\phi = \frac{1-\sqrt{5}}{2}$ are the *Golden Ratios*.

Proof by Strong Induction

Proof:

Base Case) First show that the formula holds for $n = 0$.

Left-hand-side

$$F(0) = 0$$

$=$

Right-hand-side

$$\frac{\Phi^0 - \phi^0}{\sqrt{5}} = 0$$

Proof by Strong Induction

Proof:

Base Case) We must show also that the formula holds for $n = 1$.

Left-hand-side		Right-hand-side
$F(1) = 1$	$=$	$\dfrac{\Phi^1 - \phi^1}{\sqrt{5}} = 1$

Proof by Strong Induction

Inductive Case) Assume that

(I.H.) $F(m) = \dfrac{\Phi^m - \phi^m}{\sqrt{5}}$ for all $m \leq k$ for some $k \geq 1$

We then have that for $n = (k+1)$

Proof by Strong Induction

$$
\begin{aligned}
F(k+1) &= F(k) + F(k-1) \\
&= \left(\frac{\Phi^k - \phi^k}{\sqrt{5}}\right) + \left(\frac{\Phi^{k-1} - \phi^{k-1}}{\sqrt{5}}\right) \\
&= \left(\frac{\Phi^k - \phi^k + \Phi^{k-1} - \phi^{k-1}}{\sqrt{5}}\right) \\
&= \frac{(\Phi+1)\,\Phi^{(k-1)} - (\phi+1)\,\phi^{(k-1)}}{\sqrt{5}} \\
&= \frac{(\Phi^2)\,\Phi^{(k-1)} - (\phi^2)\,\phi^{(k-1)}}{\sqrt{5}} \\
&= \frac{\Phi^{(k+1)} - \phi^{(k+1)}}{\sqrt{5}}
\end{aligned}
$$

Q.E.D.

End of Slides!