5CCS2FC2: Foundations of Computing II

Divide-and-Conquer and Recursion

Week 3

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The Master Theorem

The Master Theorem

Theorem (The Master Theorem) Let $T(n)$ be a *monotonically increasing* recurrence relation such that

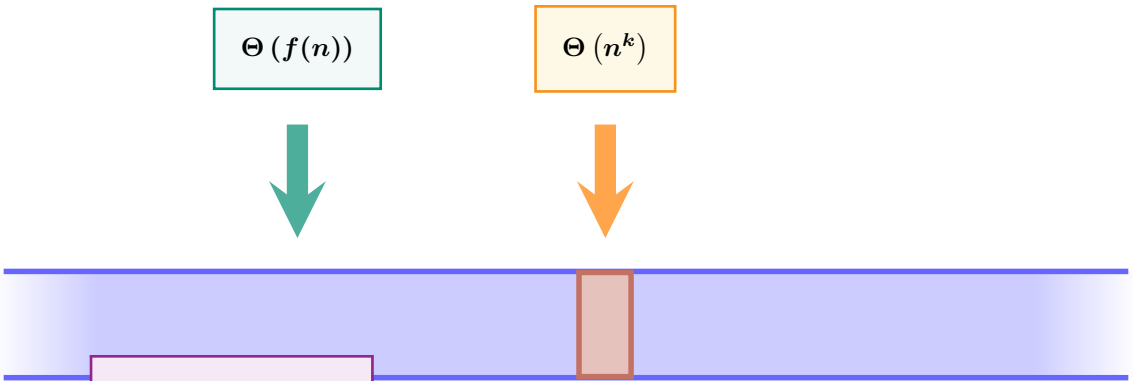
$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

for some constants $a \geq 1$, $b \geq 2$. Then

$$T(n) = \begin{cases} \Theta\left(n^k\right) & \text{if } f(n) \in O(n^{k-\varepsilon}) \\ \Theta\left(n^k \log_2 n\right) & \text{if } f(n) \in \Theta(n^k) \\ \Theta\left(f(n)\right) & \text{if } f(n) \in \Omega(n^{k+\varepsilon}) \text{ (†)} \end{cases}$$

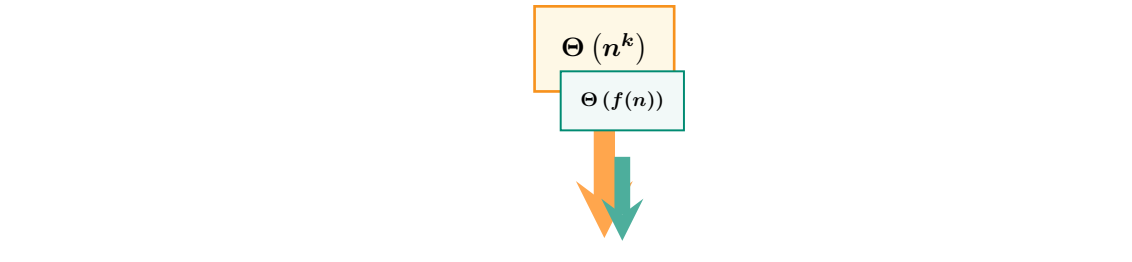
for some $\varepsilon > 0$, where $k = \log_b a$.

The Master Theorem



Case 1

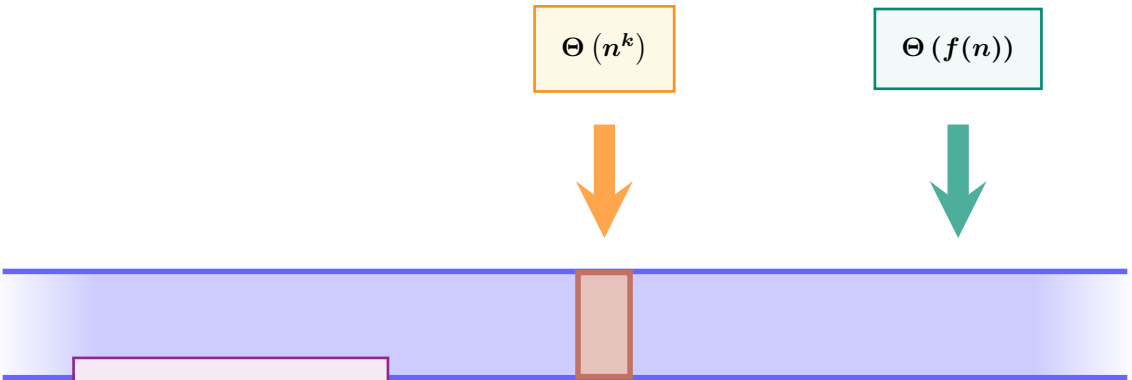
$$T(n) = \Theta(n^k)$$



Case 2

$$T(n) = \Theta\left(n^k \log_2 n\right)$$

The Master Theorem



Case 3

$$T(n) = \Theta\left(f(n)\right)$$

The Master Theorem : Some Examples

Theorem Let $T(n)$ be a recurrence relation defined by

$$T(1) = 1, \qquad T(n) = 9\,T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + \sqrt{(n+1)^5}$$

for all $n > 1$. Calculate the growth rate of $T(n)$.

The Master Theorem : Some Examples

Proof:

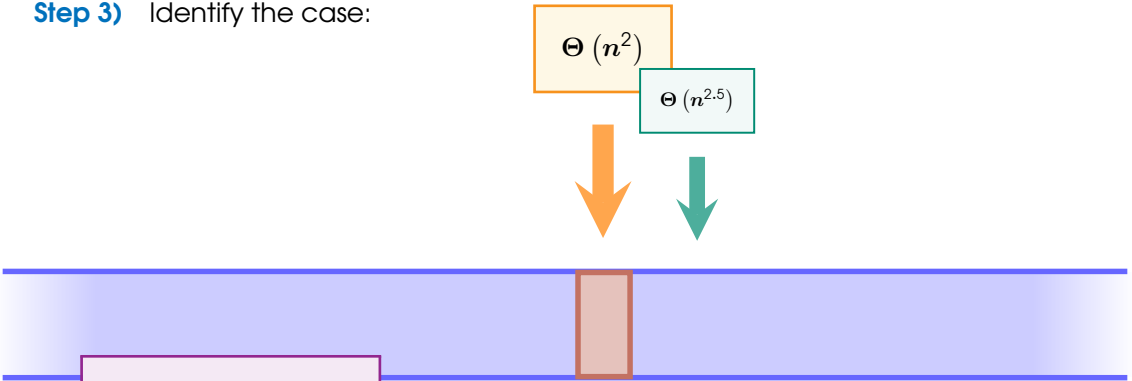
Step 1) Identify the parameters:

$$a = 9, \quad b = 3, \quad \text{therefore} \quad k = \log_3(9) = 2$$

Step 2) Identify the growth rate of $f(n)$:

$$f(n) = \sqrt{(n+1)^5} \in \Theta(n^{2.5})$$

Step 3) Identify the case:



The Master Theorem : Some Examples

Theorem Let $T(n)$ be a recurrence relation defined by

$$T(1) = 1, \qquad T(n) = 2\,T\left(\left\lceil \frac{n}{2} \right\rceil\right) + n$$

for all $n > 1$. Calculate the growth rate of $T(n)$.

Proof:

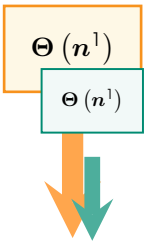
Step 1) Identify the parameters:

$$a = 2, \quad b = 2, \quad \text{therefore} \quad k = \log_2(2) = 1$$

Step 2) Identify the growth rate of $f(n)$:

$$f(n) = n \in \Theta(n^1)$$

Step 3) Identify the case:



Case 2

$$T(n) = \Theta\left(n \log_2(n)\right)$$

The Master Theorem : Some Examples

Theorem Let $T(n)$ be a recurrence relation defined by

$$T(1) = 1, \qquad T(n) = 2\,T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + \log_2(5n^2)$$

for all $n > 1$. Calculate the growth rate of $T(n)$.

Proof:

Step 1) Identify the parameters:

$$a = 2, \quad b = 3, \quad \text{therefore} \quad k = \log_3(2) \approx 1.5849$$

Step 2) Identify the growth rate of $f(n)$:

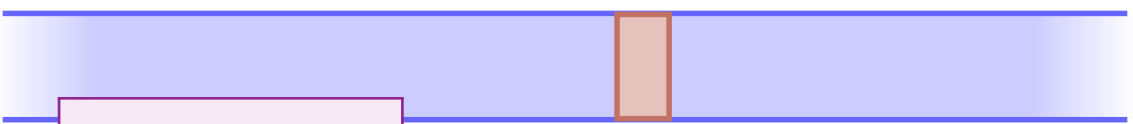
$$f(n) \;=\; \log_2(5n^2) \;\in\; \Theta\big(\log_2 n\big)$$

The Master Theorem : Some Examples

Step 3) Identify the case:

$\Theta\left(n^{\log_3(2)}\right)$

$\Theta\left(\log_2(n)\right)$



Case 1

$$T(n) = \Theta\left(n^{\log_3(2)}\right)$$

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