5CCS2FC2: Foundations of Computing II

Approximation Algorithms & the Travelling Salesman Problem

Week 7

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Proving Approximation Ratios

Proving Approximation Ratios

Theorem The 2OPT algorithm has an approximation ratio of $\boldsymbol{R} = 2$

Proof:

Step 1) Let $\boldsymbol{H}_{\text{approx}}$ and $\boldsymbol{H}_{\text{global}}$ denote the following cycles

$$\begin{aligned} \boldsymbol{H}_{\text{approx}} &= \text{Local optimal found by 2OPT algorithm} \\ \boldsymbol{H}_{\text{global}} &= \text{Global optimal Hamiltonian cycle} \end{aligned}$$

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Theorem The 2OPT algorithm has an approximation ratio of $\boldsymbol{R} = 2$

Proof:

Step 1) Let $\boldsymbol{H}_{\text{approx}}$ and $\boldsymbol{H}_{\text{global}}$ denote the following cycles

Want to show that

$$\text{length\_of}\,(\boldsymbol{H}_{\text{approx}}) \;\leq\; 2 \times \text{length\_of}\,(\boldsymbol{H}_{\text{global}})$$

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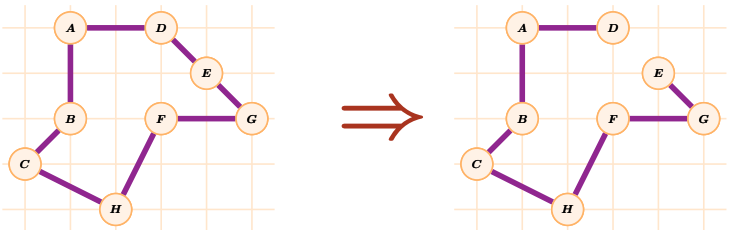
Step 2) Remove one edge from H_{global} to create a path T,

$$\text{length\_of}\,(T) \;\leq\; \text{length\_of}\,(H_{\text{global}})$$



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Step 3) Let T_0 denote the following **minimum spanning tree** computed as the first step of the 2OPT algorithm,

$$T_0 = \text{Minimum spanning tree}$$

$$\text{length\_of}\,(T_0) \leq \text{length\_of}\,(T)$$

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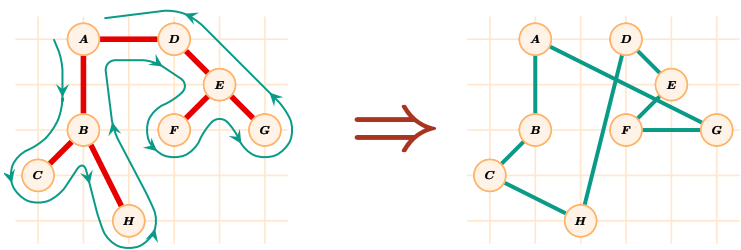
Step 4) The length of the ***pre-order traversal*** $\boldsymbol{H}_0$ is at most twice the length of the spanning tree,

$$\text{length\_of}\,(\boldsymbol{H}_0) \;\leq\; 2 \times \text{length\_of}\,(\boldsymbol{T}_0)$$



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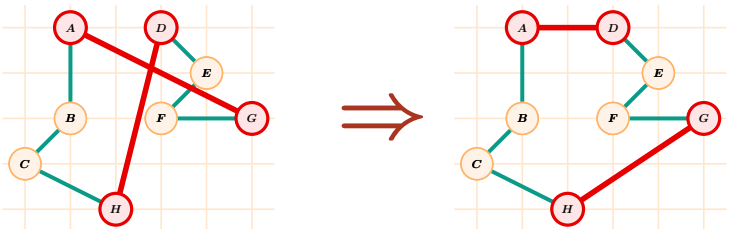
Step 5) Every application of the 2OPT swap decreases the length of the cycle, so that

$$\text{length\_of}\,(\boldsymbol{H}_{\text{approx}}) \;\leq\; \ldots \;\leq\; \text{length\_of}\,(\boldsymbol{H}_1) \;\leq\; \text{length\_of}\,(\boldsymbol{H}_0)$$



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Step 6) Therefore

$$
\begin{aligned}
\text{length\_of}\,(\boldsymbol{H}_{\textbf{approx}}) &\leq \text{length\_of}\,(\boldsymbol{H}_0) \\
&\leq 2 \times \text{length\_of}\,(\boldsymbol{T}_0) \\
&\leq 2 \times \text{length\_of}\,(\boldsymbol{T}) \\
&\leq 2 \times \text{length\_of}\,(\boldsymbol{H}_{\textbf{global}})
\end{aligned}
$$

Q.E.D.

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Theorem TSP is 2-approximable.

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Theorem (Christofides '76) TSP is $\frac{3}{2}$-approximable.

End of Slides!