5CCS2FC2: Foundations of Computing II

Approximation Algorithms & the Travelling Salesman Problem

Week 7

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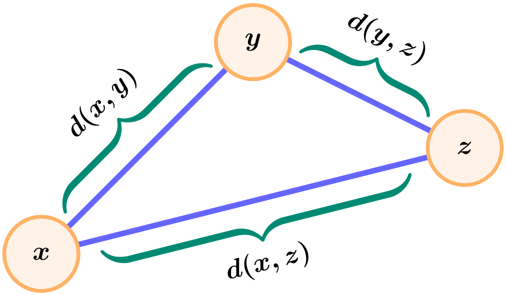
Unapproximable Instances of the Travelling Salesman Problem

Unapproximable Instances of TSP

The Triangle Inequality

$$d(x,y) + d(y,z) \;\geq\; d(x,z)$$



Unapproximable Instances of TSP

> **Theorem** The Travelling Salesman Problem **TSP** is *unapproximable* whenever d does not satisfy the **triangle inequality**.

Proof:

Step 1) Suppose, for contradiction, that there is a **polynomial time** approximate algorithm for **TSP** whose approximation ratio is R.

$$\text{length\_of}\,(H_{\text{approx}}) \;\leq\; R \times \text{length\_of}\,(H_{\text{global}})$$

Unapproximable Instances of TSP

Step 2) Consider a instance of the **Hamiltonian Cycle** problem

$$\boxed{G = (V, E)}$$

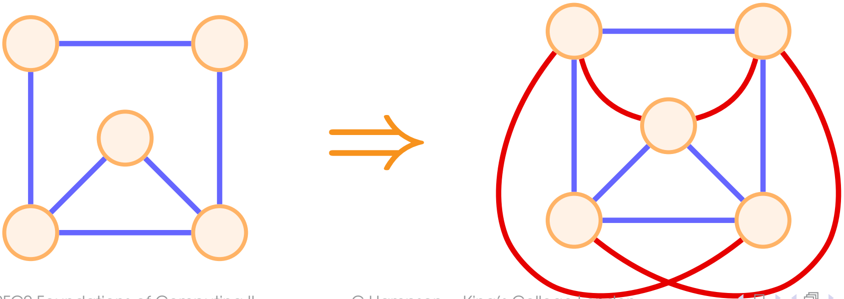
Aim

Construct an instance of **TSP** that has a *"short"* approximate solution if and only if G has a Hamiltonian cycle.

Unapproximable Instances of TSP

Step 3) Construct a complete weighted graph $G' = (V, d)$ by setting

$$d(x,y) = \begin{cases} 1 & \text{if } (x,y) \in E \\ nR+1 & \text{if } (x,y) \notin E \end{cases}$$



Unapproximable Instances of TSP

Step 4) Let H_{approx} be the cycle returned by the **approximate algorithm**. Then

$$\text{length\_of}\,(H_{\text{approx}}) \;\leq\; nR \quad \Longleftrightarrow \quad G \text{ has a Hamiltonian cycle}$$

Unapproximable Instances of TSP

Left-to-Right) Suppose that H_{approx} is a *'short'* **TSP** cycle

$$\text{length\_of}\,(H_{\text{approx}}) \;\leq\; nR$$

Then H_{approx} must use only *'short edges'* of length 1.

$\Longrightarrow$ H_{approx} must be a Hamiltonian cycle of G

Unapproximable Instances of TSP

Right-to-Left) Suppose that G has a Hamiltonian cycle. Then there is a **TSP** path H_{global} that uses only 'short' edges.

$$\boxed{\text{length\_of}\,(H_{\text{global}}) \;=\; \underbrace{1+1+\cdots+1}_{n \text{ vertices}} \;=\; n}$$

But we assumed that

$$\boxed{\text{length\_of}\,(H_{\text{approx}}) \;\leq\; R \times \text{length\_of}\,(H_{\text{global}})}$$

Unapproximable Instances of TSP

Right-to-Left) Suppose that G has a Hamiltonian cycle. Then there is a **TSP** path $H_{\textbf{global}}$ that uses only 'short' edges.

$$\text{length\_of}\,(H_{\textbf{global}}) \;=\; \underbrace{1+1+\cdots+1}_{n \text{ vertices}} \;=\; n$$

But we assumed that

$$\Longrightarrow \quad \text{length\_of}\,(H_{\textbf{approx}}) \;\leq\; R \times n$$

Step 5) Hence, we can use our hypothesized approximation algorithm to give an exact **deterministic poly-time algorithm** for the Hamiltonian cycle problem!

FAST-TSP (G):

1. $n \leftarrow |V|$, where $G = (V, E)$
1. Contruct the complete weighted graph G'
2. Let H_{approx} be result of the R-approximation algorithm
3. **If** length_of $(H_{\text{approx}}) \leq nR$:
    4. **Return True**
6. **Else**
    4. **Return False**

Unapproximable Instances of TSP

Step 6) Assuming that **P** $\neq$ **NP** we must conclude that the the general case of the Travelling Salesman Problem is **unapproximable**.

Q.E.D.

Unapproximable Instances of TSP

Routing to minimize **DISTANCE** is approximable.

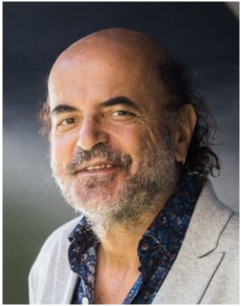
Routing to minimize **DURATION** is not!

HAMILTONIAN $\leq_p$ **TSP**$_{dec}$

Unapproximable Instances of TSP

"The Travelling Salesman Problem is not a problem; It's an addiction!"

Christos Papadimitriou



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