5CCS2FC2: Foundations of Computing II

Linear Programming

Week 8

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Linear Programming

- **Linear Programming**
  - **Linear Objective Function** The quantity to be maximised / minimised,
  - **Linear Constraints** Set of linear inequalities restricting the 'feasible' solutions.

$$
\begin{array}{rrcl}
\text{Maximise:} & 2x + 3y & & \\
\text{Subject to:} & 3x + 2y & \leq & 15 \\
 & 2y - x & \leq & 5 \\
 & x + 2y & \leq & 7
\end{array}
$$

(there may be a unique solution, infinitely many solutions or no solution)

- **Matrix notation**
  - Linear constraints can be represented more succinctly as a **matrix-vector** equation

$$\begin{bmatrix} 3 & 2 \\ -1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 15 \\ 5 \\ 7 \end{bmatrix}$$

$$A \qquad \vec{x} \quad \leq \quad \vec{b}$$

$$\boxed{A\vec{x} \leq \vec{b}}$$

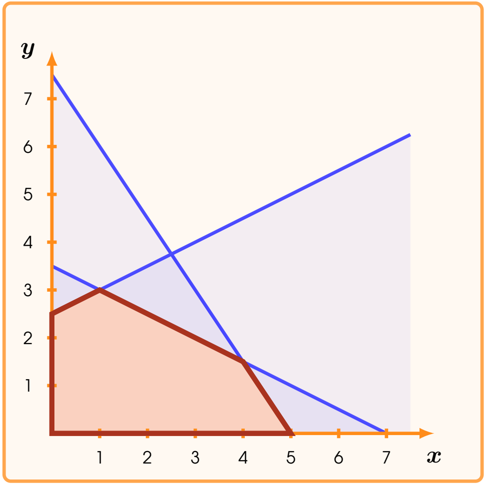
Linear Programming LP

Input) — A set of linear constraints $A\vec{x} \leq \vec{b}$,

— A linear objective function $f : \mathbb{R}^n \to \mathbb{R}$

Output) A solution $\vec{x} \in (\mathbb{R}^{\geq 0})^n$ such that **(i)** $\vec{x}$ satisfies the constraints $A\vec{x} \leq \vec{b}$ and **(ii)** $f(\vec{x})$ is maximal.

Linear Programming



$$
\begin{aligned}
3x + 2y &\leq 15 \\
2y - x &\leq 5 \\
x + 2y &\leq 7
\end{aligned}
$$

- **Feasible Region**
  - The **Feasible Region** is the set of all possible solutions satisfying the linear constraints.

$$\boxed{R = \{\vec{x} \in \mathbb{R}^n : \vec{x} \text{ is a solution}\}}$$

  - The (set of) **optimal solution(s)**, for a given objective function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$\boxed{\{\vec{x} \in R \;:\; \forall \vec{y} \in R\big(f(\vec{x}) \geq f(\vec{y})\big)\} \;\subseteq\; R}$$

Example

Linear Programming Example

- **Balancing Energy Demands**

		Natural Gas	Coal	Solar	Wind	Nuclear
CO_2 Emissions	(tonne/GWh)	500	1000	0	0	0
Cost	(GBP/kWh)	40	80	22	22	120
Capacity	(GW)	32	8	13.5	24	10

Linear Programming Example

- **Balancing Energy Demands**

Step 1) For each mode of energy production introduce a new **numerical variable** G, C, S, W, N to denote the **energy production (kWh)**.

Step 2) We must ensure that each mode does not exceed its **capacity**

$$
\begin{array}{lllllll}
G & \leq & 32 & & C & \leq & 8 \\
S & \leq & 13.5 & & W & \leq & 24 \\
N & \leq & 10 & & & &
\end{array}
$$

Linear Programming Example

- **Balancing Energy Demands**

Step 3) The want the total energy production must meet the **energy demands**

$$\boxed{G + C + S + W + N \geq \text{total energy demands}}$$

Step 4) We also don't want to exceed **target CO_2 emissions**

$$\boxed{0.5\,G + 1\,C \leq \text{target } CO_2 \text{ emissions}}$$

Linear Programming Example

- **Balancing Energy Demands**

  Step 5) Finally, we want to minimise the **total cost**

$$\text{Minimize} \quad 40\,\boldsymbol{G} + 80\,\boldsymbol{C} + 22\,\boldsymbol{S} + 22\,\boldsymbol{W} + 120\,\boldsymbol{N}$$

End of Slides!