5CCS2FC2: Foundations of Computing II

Linear Programming

Week 8

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The Simplex Method

The Simplex Method

- **Dantzig's Simplex Method**
  - Proposed in 1946-47 by G.Dantig following his work on **operations research** for the US Air Force during WWII
  - Named after a type of n-**dimensional polygon** called a **Simplex**, due to the shape of the feasible region
  - Dantzig is widely cited as the inspiration for Matt Damon's character, Will, in the film **Good Will Hunting**

- **Slack Variables**
  - We can introduce a **slack variable** for each constraint, turning each *inequality* into an *equation* called the **slack form**

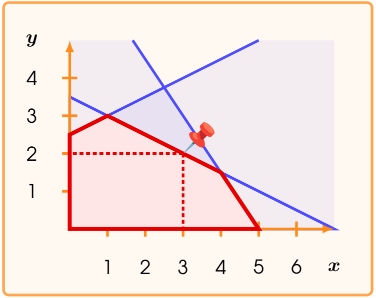
$$\begin{aligned} 3x + 2y &\leq 15 \\ 2y - x &\leq 5 \\ x + 2y &\leq 7 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 3x + 2y + s_1 &= 15 \\ 2y - x + s_2 &= 5 \\ x + 2y + s_3 &= 7 \end{aligned}$$

- **Slack Variables**
  - For every choice of the **independent variables** x and y, we can caluclate the value of the **dependent** slack variables s_1, s_2, s_3.

$$\begin{aligned} s_1 &= 15 - 3x - 2y \\ s_2 &= 5 + x - 2y \\ s_3 &= 7 - x - 2y \end{aligned}$$

- **Slack Variables**
  - For every choice of the **independent variables** x and y, we can calculate the value of the **dependent** slack variables s_1, s_2, s_3.

$$\begin{aligned} 2 &= 15 - 3(3) - 2(2) \\ 4 &= 5 + (3) - 2(2) \\ 0 &= 7 - (3) - 2(2) \end{aligned}$$



- **The Cost Variable**
  - We can treat the objective function as an equation for some new **dependent** variable C called the **cost equation**

$$\boxed{C = 2x + 3y}$$

Introducing Tableaux

- **Tableaux**
  - A **Tableau** (*plural* **Tableaux**) is a matrix representation of a system of linear equations:

Linear Constraints

$$\begin{aligned} 3x + 2y + s_1 &= 15 \\ -x + 2y + s_2 &= 5 \\ x + 2y + s_3 &= 7 \end{aligned}$$

Cost Equation

$$C = 2x + 3y$$

$$\left\{ \begin{bmatrix} 3x + 2y + 1s_1 & & & & = 15 \\ \text{-}1x + 2y & + 1s_2 & & & = 5 \\ 1x + 2y & & + 1s_3 & & = 7 \\ \text{-}2x + \text{-}3y & & & + 1C & = 0 \end{bmatrix} \right.$$

- **The Simplex Method**

Step 1) Construct the **initial tableau** from the slack form of the linear program, together with the equation for the cost C.

$$
\begin{array}{c}
\begin{array}{cccccccc} \;\;x & \;\;\;y & \;\;s_1 & \;\;s_2 & \;\;s_3 & \;\;C & & \end{array} \\
\left[\begin{array}{cccccc:c}
3 & 2 & 1 & 0 & 0 & 0 & 15 \\
-1 & 2 & 0 & 1 & 0 & 0 & 5 \\
1 & 2 & 0 & 0 & 1 & 0 & 7 \\
\hdashline
-2 & -3 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

- **The Simplex Method**

Step 2) Identify the column with the **most negative** coefficient in the final row, corresponding to the **objective function**.

'pivot' column

$$
\begin{array}{cccccccc}
x & y & s_1 & s_2 & s_3 & C & & \\
\end{array}
$$

$$
\left[\begin{array}{cccccc:c}
3 & 2 & 1 & 0 & 0 & 0 & 15 \\
-1 & 2 & 0 & 1 & 0 & 0 & 5 \\
1 & 2 & 0 & 0 & 1 & 0 & 7 \\
\hdashline
-2 & -3 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
$$

- **The Simplex Method (cont.)**

Step 3) Calculate the **row quotients** by dividing each of the entries in the final column by the entries in the pivot column

$$
\begin{array}{c}
\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & C & \end{array} \\
\left[\begin{array}{cccccc|c}
3 & 2 & 1 & 0 & 0 & 0 & 15 \\
-1 & 2 & 0 & 1 & 0 & 0 & 5 \\
1 & 2 & 0 & 0 & 1 & 0 & 7 \\
\hline
-2 & -3 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\begin{array}{l}
\leftarrow \frac{15}{2} \\
\leftarrow \frac{5}{2} \\
\leftarrow \frac{7}{2} \\

\end{array}
$$

- **The Simplex Method (cont.)**

Step 4) Identify the row with the **smallest** row quotient where *both* the numerator and denominator are **positive**

$$
\begin{array}{c}
\begin{array}{cccccccc} x & y & s_1 & s_2 & s_3 & C & & \end{array} \\
\left[\begin{array}{cccccc:c}
3 & 2 & 1 & 0 & 0 & 0 & 15 \\
-1 & 2 & 0 & 1 & 0 & 0 & 5 \\
1 & 2 & 0 & 0 & 1 & 0 & 7 \\
\hdashline
-2 & -3 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\begin{array}{l} \\ \\ \leftarrow \text{'pivot' row} \\ \\ \\ \end{array}
$$

- **The Simplex Method (cont.)**

Step 5) The **'pivot' value** is the value in the 'pivot' column and in the 'pivot' row.

'pivot' value

$$\begin{array}{c}
\begin{array}{cccccccc} x & y & s_1 & s_2 & s_3 & C & & \end{array} \\
\left[\begin{array}{cccccc:c}
 & 2 & 1 & 0 & 0 & 0 & 15 \\
-1 & 2 & 0 & 1 & 0 & 0 & 5 \\
1 & 2 & 0 & 0 & 1 & 0 & 7 \\
\hdashline
-2 & -3 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

- **The Simplex Method (cont.)**

  Step 5) Apply the following **row transformation**

  - **Pivot row**

$$\boxed{\boldsymbol{R}_{\text{pivot}} \leftarrow \left(\frac{1}{\text{current pivot value}}\right) \times \boldsymbol{R}_{\text{pivot}}}$$

  - **All non-pivot rows**

$$\boxed{\boldsymbol{R}_i \leftarrow \boldsymbol{R}_i - \left(\frac{\text{current } \boldsymbol{R}_i \text{ value}}{\text{current pivot value}}\right) \times \boldsymbol{R}_{\text{pivot}}}$$

- **The Simplex Method (cont.)**

Step 5) Applying these **row transformation** gives

$$
\begin{array}{cccccccc}
x & y & s_1 & s_2 & s_3 & C & & \\
\end{array}
$$

$$
\left[\begin{array}{cccccc:c}
4 & 0 & 1 & -1 & 0 & 0 & 10 \\
-0.5 & 1 & 0 & 0.5 & 0 & 0 & 2.5 \\
2 & 0 & 0 & -1 & 1 & 0 & 2 \\
\hdashline
-3.5 & 0 & 0 & 1.5 & 0 & 1 & 7.5
\end{array}\right]
$$

Round 1

$$R_1 \leftarrow \left(R_1 - \frac{2}{2}R_2\right)$$

$$R_2 \leftarrow \frac{1}{2}R_2$$

$$R_3 \leftarrow \left(R_3 - \frac{2}{2}R_2\right)$$

$$R_4 \leftarrow \left(R_4 - \frac{3}{2}R_2\right)$$

- **The Simplex Method (cont.)**

Step 6) Repeat until you run out of valid pivot columns

$$
\begin{array}{cccccccc}
x & y & s_1 & s_2 & s_3 & C & & \\
\end{array}
$$

$$
\left[\begin{array}{cccccc:c}
0 & 0 & 1 & 1 & -2 & 0 & 6 \\
0 & 1 & 0 & 0.25 & 0.25 & 0 & 3 \\
1 & 0 & 0 & -0.5 & 0.5 & 0 & 1 \\
\hdashline
0 & 0 & 0 & -0.25 & 1.75 & 1 & 11
\end{array}\right]
$$

Round 2

$$R_1 \leftarrow \left(R_1 - \frac{4}{2} R_3\right)$$

$$R_2 \leftarrow \left(R_3 - \frac{-0.5}{2} R_3\right)$$

$$R_3 \leftarrow \frac{1}{2} R_3$$

$$R_4 \leftarrow \left(R_4 - \frac{-3.5}{2} R_3\right)$$

- **The Simplex Method (cont.)**

Step 6) Repeat until you run out of valid pivot columns

$$
\begin{array}{cccccccc}
x & y & s_1 & s_2 & s_3 & C & & \\
\end{array}
$$
$$
\left[\begin{array}{cccccc:c}
0 & 0 & 1 & 1 & -2 & 0 & 6 \\
0 & 1 & -0.25 & 0 & 0.75 & 0 & 1.5 \\
1 & 0 & 0.5 & 0 & -0.5 & 0 & 4 \\
\hdashline
0 & 0 & 0.25 & 0 & 2.25 & 1 & 12.5
\end{array}\right]
$$

Round 3

$$R_1 \leftarrow \frac{1}{1} R_1$$

$$R_2 \leftarrow \left(R_2 - \frac{0.25}{1} R_1 \right)$$

$$R_3 \leftarrow \left(R_3 - \frac{-0.5}{1} R_1 \right)$$

$$R_4 \leftarrow \left(R_4 - \frac{-0.25}{2} R_1 \right)$$

- **The Simplex Method (cont.)**

Step 7) Read off the optimal solution

$$\begin{array}{cccccc} x & y & s_1 & s_2 & s_3 & C \end{array}$$

$$\left[\begin{array}{c} \\ \\ \\ 0.25s_1 \quad + \quad 2.25s_3 + C = 12.5 \end{array} \right]$$

Optimal Solution

$$\begin{aligned} x &= \text{—} \\ y &= \text{—} \\ s_1 &= 0 \\ s_2 &= \text{—} \\ s_3 &= 0 \\ C &= 12.5 \end{aligned}$$

- **The Simplex Method (cont.)**

Step 7) Read off the optimal solution

$$
\begin{array}{c}
\begin{array}{cccccc} x & y & s_1 & s_2 & s_3 & C \end{array} \\
\left[\begin{array}{cccccc}
 & & & & & \\
x & + & 0.5s_1 & + & \text{-}0.5s_3 & = 4 \\
 & & & & &
\end{array}\right]
\end{array}
$$

Optimal Solution

$$
\begin{aligned}
x &= 4 \\
y &= \text{—} \\
s_1 &= 0 \\
s_2 &= \text{—} \\
s_3 &= 0 \\
C &= 12.5
\end{aligned}
$$

- **The Simplex Method (cont.)**

Step 7) Read off the optimal solution

$$\begin{array}{cccccc} x & y & s_1 & s_2 & s_3 & C \end{array}$$
$$\left[\begin{array}{c} \\ y + -0.25 s_1 \; + \; 0.75 s_3 \; = \; 1.5 \\ \\ \end{array}\right]$$

Optimal Solution

$$\begin{aligned} x &= 4 \\ y &= 1.5 \\ s_1 &= 0 \\ s_2 &= - \\ s_3 &= 0 \\ C &= 12.5 \end{aligned}$$

- **The Simplex Method (cont.)**

Step 7) Read off the optimal solution

$$
\begin{array}{cccccc}
x & y & s_1 & s_2 & s_3 & C \\
\end{array}
$$

$$
\left[\begin{array}{c}
s_1 + s_2 + \text{-}2s_3 \quad = \quad 6 \\
\\
\\
\end{array}\right]
$$

Optimal Solution

$$
\begin{aligned}
x &= 4 \\
y &= 1.5 \\
s_1 &= 0 \\
s_2 &= 6 \\
s_3 &= 0 \\
C &= 12.5
\end{aligned}
$$

The Simplex Method

Theorem (Klee–Minty '72) The worst-case termination time for the Simplex Method exponential.

$$
\begin{array}{lrcl}
\text{Maximise:} & 2^n x_0 + 2^{n-1}x_1 + \cdots + 2x_{n-1} + x_n & & \\
\text{Subject to:} & x_0 & \leq & 5 \\
 & 4x_0 + x_1 & \leq & 25 \\
 & 8x_0 + 4x_1 + x_3 & \leq & 125 \\
 & \vdots & & \vdots \\
 & 2^{n+1}x_0 + 2^n x_1 + \cdots + 4x_{n-1} + x_n & \leq & 5^{n+1} \\
 & x_0, x_1, \ldots, x_n & \geq & 0
\end{array}
$$

The Simplex Method

Theorem (Khachiyan '79) Linear Programming is solvable in Polynomial Time.

End of Slides!