5CCS2FC2: Foundations of Computing II

Probability, Expectation & Probabilistic Complexity Classes

Week 9

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Recap of Probability, Events, and Random Variables

Probability Space

- **Sample Spaces and Probability Measures**
  - A **sample space** is a set of all possible outcomes from some random process

$$\boxed{S = \{\text{ all possible outcomes }\}}$$

$$\boxed{S = \{s_1, s_2, \ldots, s_n\}}$$

  - A **probability measure** on S is a function $\boxed{\textbf{Prob}: S \to [0, 1]}$

Probability Space

- **Events**
  - An **event** is any subset

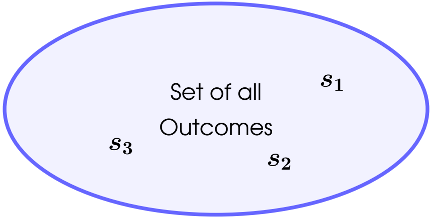
$$\boxed{E \subseteq S} \qquad \boxed{E \in \mathcal{P}(S)}$$

  - We extend the **probability measure** to events

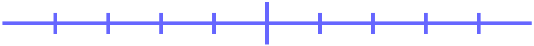
$$\boxed{\mathsf{Prob} : \mathcal{P}(S) \to [0, 1]}$$

$$\boxed{\mathsf{Prob}(E) = \sum_{s \in E} \mathsf{Prob}(s)}$$

- **Random Variables**
  - A **random variable** on S is a *measurement* of a random outcome $s \in S$



$$X : S \to \mathbb{R}$$



Random Variables

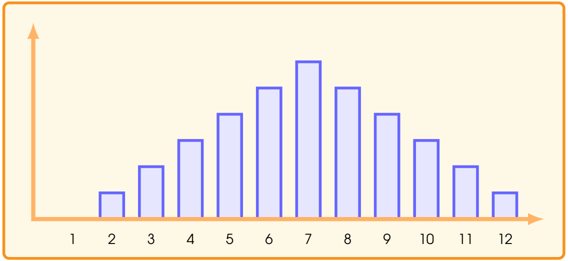
- **Probabilities of Random Variables**
  - The probability that a **random variable** X takes the value $k \in \mathbb{R}$ is given by

$$\boxed{\textbf{Prob}\,(X = k) \;=\; \textbf{Prob}\left(\{s \in S \;:\; X(s) = k\}\right)}$$

  - Similarly,

$$\boxed{\textbf{Prob}\,(X \leq k) \;=\; \textbf{Prob}\left(\{s \in S \;:\; X(s) \leq k\}\right)}$$

- **Probability Mass Function (p.m.f.)**
  - The **probability mass function** of a random variable X is a function which says how likely a given value is to appear as the measurement of a random event.



$$p_X(k) = \textbf{Prob}\,(X = k)$$

Expectation and Variance

Expectation of a Random Variable

- **Expectation / (Mean) Average**
  - The **expectation** of a random variable X is the *weighted average* of the possible values of X

$$\boxed{\mathbf{E}[X] = \sum_k k \cdot p_X(k)}$$

- **Expectation / (Mean) Average**
  - The **expectation** of a random variable X is the *weighted average* of the possible values of X

k	1	2	3	4	5	6
$p_X(k)$	0.15	0.3	0.15	0.1	0.05	0.25
$k \cdot p_X(k)$	0.15	0.6	0.45	0.4	0.25	1.5

$$\begin{array}{rr} & 0.15 \\ + & 0.6 \\ + & 0.45 \\ + & 0.4 \\ + & 0.25 \\ + & 1.5 \\ \hline & 3.35 \end{array}$$

$$\mathbf{E}[X] = 3.35$$

Properties of Expectation

Theorem Let X and Y be any two random variable and let $a, b \in \mathbb{R}$ be real numbers. Then:

(i) $\mathsf{E}[aX + b] = a\,\mathsf{E}[X] + b$,

(ii) $\mathsf{E}[X + Y] = \mathsf{E}[X] + \mathsf{E}[Y]$.

Properties of Expectation

Theorem (Markov's Inequality) Let X be a random variable with expectation $\mathbf{E}[X]$. Then

$$\mathbf{Prob}\left(X \geq a \cdot \mathbf{E}[X]\right) \leq \frac{1}{a}$$

for any $a > 0$.

Variance of a Random Variable

- **Variance**
  - The **variance** of a random variable X is a measure of the *expected deviation* from the average $\mu = \mathbf{E}[X]$

Attempt 1

$$
\begin{aligned}
\mathbf{Var}[X] &\stackrel{?}{=} \mathbf{E}\left[\text{differences between } X \text{ and } \mu\right] \\
&= \mathbf{E}[X - \mu] = \text{always zero!} \quad ✗
\end{aligned}
$$

Variance of a Random Variable

- **Variance**
  - The **variance** of a random variable X is a measure of the *expected deviation* from the average $\mu = \mathbf{E}[X]$

Attempt 2

$$\begin{aligned} \mathbf{Var}[X] &\stackrel{?}{=} \mathbf{E}\,[\textit{squared}\text{ differences between } X \text{ and } \mu] \\ &= \mathbf{E}\left[(X-\mu)^2\right] \\ &= \text{a better measure of deviation} \quad ✓ \end{aligned}$$

Variance of a Random Variable

Theorem Let X be a random variable. Then, we have that

$$\mathbf{Var}\left[X\right] = \mathbf{E}\left[X^2\right] - \mathbf{E}\left[X\right]^2$$

Variance of a Random Variable

- **Standard Deviation**
  - The **standard deviation** of a random variable X is the square-root of the variance of X,

$$\boxed{\sigma_X = \sqrt{\mathsf{Var}\,[X]}}$$

End of Slides!