5CCS2FC2: Foundations of Computing II

Probability, Expectation & Probabilistic Complexity Classes

Week 9

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Probabilistic Turing Machines

Probabilistic Turing Machines

- **Probabilistic Turing Machine (PTM)**

$$\boxed{M = \langle\, \Sigma,\ Q,\ q_{\text{init}},\ q_{\text{accept}},\ q_{\text{reject}},\ \delta \,\rangle}$$

- Alphabet $\quad \boxed{\Sigma = \{a_1, \ldots, a_k\}}$
- Finite set of control states $\quad \boxed{Q = \{q_1, \ldots, q_n\}}$
- Initial state $\quad \boxed{q_{\text{init}} \in Q}$
- Accept and Reject states $\quad \boxed{q_{\text{accept}}, q_{\text{reject}} \in Q}$

As before!

Probabilistic Turing Machines

- **Probabilistic Turing Machine (PTM)**

$$\boxed{M = \langle\, \Sigma,\ Q,\ q_{\text{init}},\ q_{\text{accept}},\ q_{\text{reject}},\ \delta \,\rangle}$$

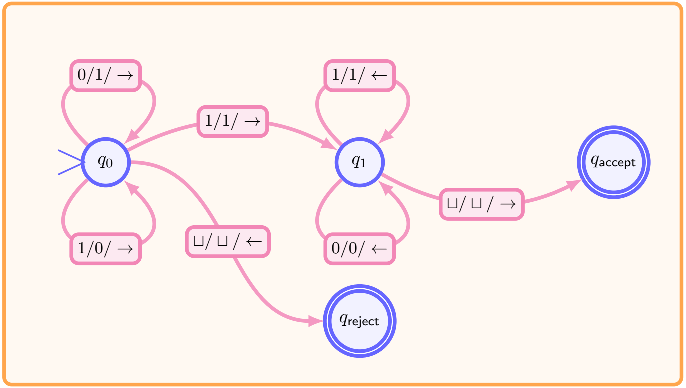
  - *Probabilistic* transition function

$\delta($ current state, read symbol $) \mapsto$ probability distribution on possible instructions

- **Sample Space**

$$\boxed{S = \mathcal{P}\,(Q \times \Sigma \cup \{\sqcup\} \times \{\leftarrow, \rightarrow\})}$$

Probabilistic Turing Machines

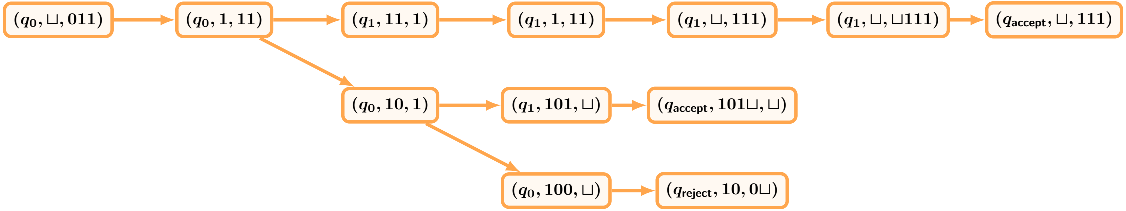


Probabilistic Turing Machines

- **Configurations and Computations**
  - The **configurations** of a PTM M are defined as for regular Turing Machines as a complete description of the machine at any given time

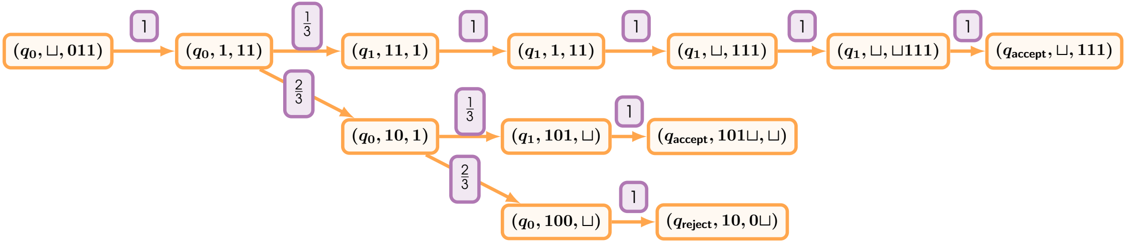
$$\boxed{\big(\,\text{current state},\ \text{left substring},\ \text{right substring}\,\big)}$$

- **Configurations and Computations**
  - The **computations** of a PTM M are all possible sequences of configurations obtained by running M on a given input

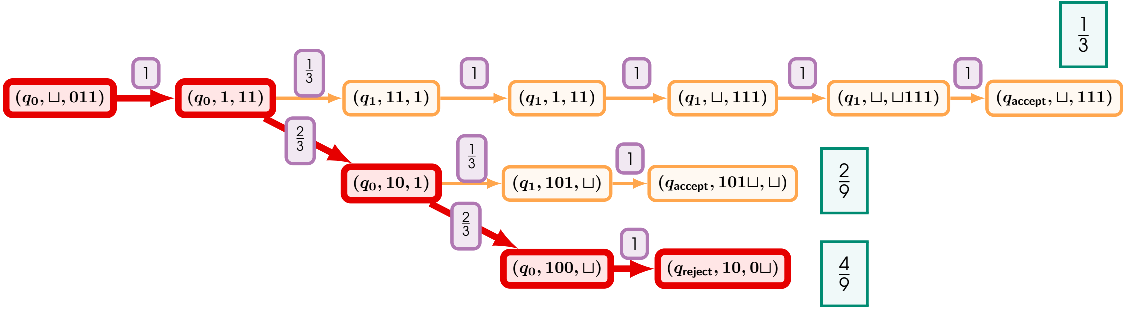


Probabilistic Turing Machines

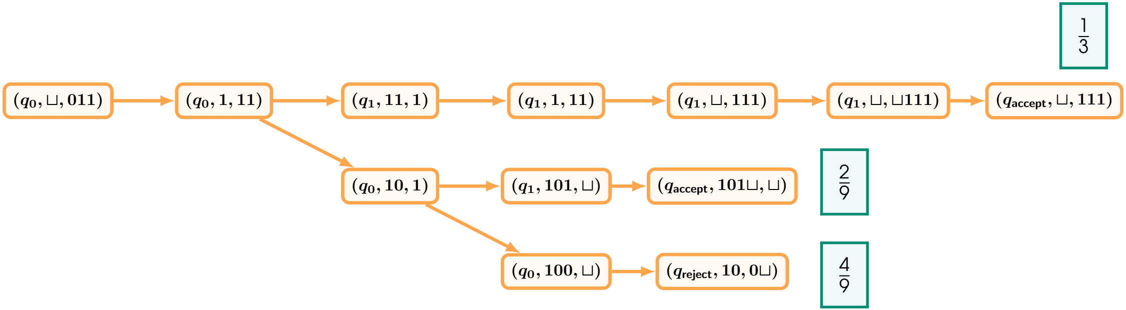
- **Configurations and Computations**
  - However, each **(branching) transition** now has as associated **probability**, stating the likelihood that the machine will apply the associated instruction.



- **Configurations and Computations**
  - We can compute the **probability** that M follows a particular branch by **multiplying the probabilities** along the branch



- **Acceptance Criteria**
  - We can compute the **probability** that a PTM M follows a partiular path by **multiplying the probabilities** along the path in the branching computation tree



Probabilistic Turing Machines

- **Acceptance Criteria**
  - We can compute the **probability** that a PTM M accepts an input word w by **multiplying the probabilities** along all paths that lead to the accept state and adding them together.

$$
\begin{aligned}
\textbf{Prob}\,(M(w) = 1) &= \textbf{Prob}\,(\text{branch 1 } \textbf{OR} \text{ branch 2 } \textbf{OR} \ldots) \\
&= \textbf{Prob}\,(\text{branch 1}) + \textbf{Prob}\,(\text{branch 2}) + \ldots \\
&= \sum_{\substack{\text{all accepting} \\ \text{branches}}} \textbf{Prob}\,(\text{accepting branch})
\end{aligned}
$$

Probabilistic Turing Machines

- **Measurements of PTMs**
  - Any **numerical property** that we can associate with a computation can be considered a **random variable**
    - The **Termination time** $T(n)$
    - The amount of **Space / Tape** used $S(n)$
    - Number of times the machine is in a **given state**, *etc.*

End of Slides!