Appendix A

Mathematical Background

This appendix collects the mathematical prerequisites, stated for reference rather than developed from scratch.

A.1 Sets, functions, correspondences

A *correspondence* $F : X \rightrightarrows Y$ assigns to each $x \in X$ a subset $F(x) \subseteq Y$ (e.g., best-response sets). F has a *closed graph* if $x^k \to x$, $y^k \to y$, $y^k \in F(x^k)$ imply $y \in F(x)$. A set $K \subseteq \mathbb{R}^m$ is *compact* iff it is closed and bounded (Heine–Borel), and *convex* if it contains the segment between any two of its points. The *simplex* $\Delta(A) = \{p \in \mathbb{R}^{|A|}_{\geq 0} : \sum_a p_a = 1\}$—the home of mixed strategies—is compact and convex.

A.2 Optimization facts

- **Weierstrass**: a continuous function on a nonempty compact set attains a maximum.
- **First-order condition**: at an interior maximum of a differentiable function, the gradient vanishes; with concavity, the FOC is also sufficient.
- **Berge's maximum theorem**: if $u(x, \theta)$ is continuous and the constraint set $C(\theta)$ is a continuous, compact-valued correspondence, then the value function is continuous and the maximizer correspondence is upper hemicontinuous with closed graph—the workhorse behind continuity of best responses.
- **Envelope theorem**: the derivative of a value function with respect to a parameter equals the partial derivative of the objective at the optimum; integral form (Milgrom–Segal) underlies Lemma 14.5.
- **Jensen's inequality**: for concave u, $\mathbb{E}[u(W)] \leq u(\mathbb{E}[W])$—the definition of risk aversion.

A.3 Probability

Finite probability spaces suffice for most of this book. Key tools: expectation and its linearity; conditional probability and **Bayes' rule**

$$\mathbb{P}(\theta \mid s) \;=\; \frac{\mathbb{P}(s \mid \theta)\,\mathbb{P}(\theta)}{\sum_{\theta'} \mathbb{P}(s \mid \theta')\,\mathbb{P}(\theta')},$$

which drives every updating argument in Chapters 10–11; independence; and for auction theory, *order statistics*: with n i.i.d. draws from F, the maximum has CDF $F(v)^n$, and for $U[0,1]$ the expected k-th highest of n is $\frac{n+1-k}{n+1}$.

A.4 Fixed points

Stated as Theorems 13.1 and 13.2. Intuition for Brouwer in one dimension: a continuous $f : [0,1] \to [0,1]$ must cross the diagonal (intermediate value theorem applied to $f(x) - x$). Kakutani generalizes to correspondences; convex-valuedness cannot be dropped (rotating the circle has no fixed point when values are allowed to be disconnected). Sperner's lemma provides the standard combinatorial proof route and also explains the computational hardness of finding fixed points (Chapter 17).

A.5 Linear programming and duality

The primal–dual pair

$$\max\{c^\top x : Ax \le b,\, x \ge 0\} \qquad \longleftrightarrow \qquad \min\{b^\top y : A^\top y \ge c,\, y \ge 0\}$$

satisfies *weak duality* (any feasible pair has $c^\top x \le b^\top y$) and *strong duality* (optimal values coincide when either problem is feasible and bounded). Complementary slackness links optimal solutions. Applications in this book: the Minimax Theorem (Chapter 12), the Bondareva–Shapley theorem (Chapter 15), and polynomial-time computation of correlated equilibria (Chapter 17).

A.6 Basic dynamical systems

For an ODE $\dot{x} = f(x)$ on the simplex: *rest points* satisfy $f(x^*) = 0$; a rest point is *Lyapunov stable* if trajectories starting nearby stay nearby, and *asymptotically stable* if they also converge to it. In one dimension, stability is read off the sign of f around the rest point; in higher dimensions, from the eigenvalues of the Jacobian or via a Lyapunov function. These suffice for the replicator analyses of Chapter 16.