Chapter 5

Mixed Strategies

Matching Pennies has no pure Nash equilibrium: any deterministic play can be exploited. The way out, foreshadowed by the practice of every poker player and penalty-kick taker, is to *randomize*.

5.1 Mixed strategies and expected payoffs

Definition 5.1: Mixed strategy

A *mixed strategy* for player i is a probability distribution $\sigma_i \in \Delta(A_i)$ over her pure actions. The *support* of σ_i, written $\operatorname{supp}(\sigma_i)$, is the set of actions played with positive probability. A profile $\sigma = (\sigma_1, \dots, \sigma_n)$ induces, via independent randomization, expected payoffs

$$u_i(\sigma) \;=\; \sum_{a \in A} \Big(\prod_{j \in N} \sigma_j(a_j) \Big)\, u_i(a).$$

By Theorem 2.2 (expected utility), players evaluate mixed profiles by these expectations. A *mixed-strategy Nash equilibrium* is defined exactly as before: σ^* such that $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$ for every player i and every mixed strategy σ_i.

5.2 The indifference principle

The single most useful tool for computing mixed equilibria:

Proposition 5.1: Indifference principle

σ^* is a Nash equilibrium if and only if, for every player i:

1. every action in $\operatorname{supp}(\sigma_i^*)$ yields the same expected payoff against σ_{-i}^*; and
2. no action outside the support yields a higher expected payoff against σ_{-i}^*.

Proof. If some action in the support were worse than another, shifting its probability to the better action would raise u_i—contradicting equilibrium. Conversely, if all support actions are equally good and no outside action is better, every mixture over the support is a best response, including σ_i^* itself. □

Key Idea: You mix to make your *opponent* indifferent

In equilibrium, a player's mixing probabilities are determined by the requirement of keeping the *other* player indifferent—not herself well-off. This is counterintuitive and powerful: your equilibrium mixture depends on your opponent's payoffs, not your own.

Example 5.1: Matching Pennies solved

Let player 1 play H with probability p and player 2 play H with probability q. Player 1 is willing to mix only if H and T give her equal expected payoffs:

$$\underbrace{q(1)+(1-q)(-1)}_{u_1(H)} = \underbrace{q(-1)+(1-q)(1)}_{u_1(T)} \implies 2q-1 = 1-2q \implies q = \tfrac{1}{2}.$$

Symmetrically $p = \frac{1}{2}$. The unique equilibrium is both mixing 50–50, with value 0 each.

Example 5.2: Battle of the Sexes: the mixed equilibrium

In Example 3.3, besides the two pure equilibria there is a mixed one. Let row play B with probability p; column mixes only if indifferent: $u_2(B) = p \cdot 1$, $u_2(F) = (1-p)\cdot 3$, so $p = \frac{3}{4}$. Symmetrically column plays B with probability $q = \frac{1}{4}$. Expected payoffs are $\frac{3}{4}$ each—*worse for both* than either pure equilibrium (3 and 1): the players miscoordinate with probability $1 - \left(\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{3}{4}\right) = \frac{5}{8}$.

Example 5.3: Hawk–Dove and mixing in populations

	Hawk	Dove
Hawk	$\frac{V-C}{2}, \frac{V-C}{2}$	$V, 0$
Dove	$0, V$	$\frac{V}{2}, \frac{V}{2}$

Two animals contest a resource of value V; fighting costs $C > V$. The symmetric mixed equilibrium has each playing Hawk with probability $p^* = V/C$. In biology this is reinterpreted: a stable *population* contains a fraction V/C of aggressive types. This reinterpretation of mixed equilibrium as population frequencies is the gateway to evolutionary game theory (Chapter 16).

5.3 Interpreting mixed strategies

Do people literally roll dice? Four standard interpretations:

1. **Deliberate randomization**: in strictly competitive settings (poker bluffs, soccer penalties, tax audits, military tactics), unpredictability is genuinely valuable and players consciously randomize.
2. **Population frequencies**: the mixture describes proportions of a population playing each pure action (Example 5.3).
3. **Beliefs**: σ_i is not what i does but what *others believe* i will do; equilibrium is a consistency condition on beliefs.
4. **Purification (Harsanyi 1973)**: each player has tiny private payoff perturbations and

plays a *pure* best response to her realized type; to outsiders this looks exactly like the mixed equilibrium. The mixed equilibrium is the limit of pure equilibria of nearby incomplete-information games.

Remark: Empirical evidence

Professional soccer penalty kicks are a celebrated test: kickers and goalkeepers choose sides with frequencies closely matching the mixed equilibrium computed from scoring rates, and their choices are serially independent, as theory requires (Palacios-Huerta 2003). Novices in labs, by contrast, alternate too much—true randomness is hard for humans.

5.4 Computing equilibria in larger games

For $2 \times n$ and small games, a practical algorithm:

1. Find all pure equilibria by best-response marking.

2. For each candidate *support profile* (subset of actions for each player), impose indifference within supports, solve the linear equations, and check: probabilities in $[0, 1]$, and no profitable deviation outside the support.

The number of support profiles grows exponentially—a first hint that computing Nash equilibria is genuinely hard (formalized in Chapter 17).

Example 5.4: A 2×3 game

	L	M	R
T	$2, 2$	$0, 3$	$1, 0$
B	$1, 0$	$2, 1$	$0, 2$

No pure equilibrium (check by marking). Suppose player 1 mixes $pT + (1-p)B$ with $p \in (0, 1)$. Player 2's expected payoffs: $u_2(L) = 2p$, $u_2(M) = 3p + (1-p) = 2p + 1$, $u_2(R) = 2(1-p)$. Since $u_2(M) > u_2(L)$ always, L is never in player 2's best responses when 1 mixes; try support $\{M, R\}$: indifference $2p + 1 = 2 - 2p$ gives $p = \frac{1}{4}$. Player 1's indifference between T and B against $qM + (1-q)R$: $0 \cdot q + 1(1-q) = 2q + 0(1-q)$, so $q = \frac{1}{3}$. Check deviations: player 2's payoff from L is $2p = \frac{1}{2} < u_2(M) = \frac{3}{2}$. Equilibrium: $\sigma_1 = (\frac{1}{4}, \frac{3}{4})$, $\sigma_2 = (0, \frac{1}{3}, \frac{2}{3})$.

Theorem 5.2: Oddness (Wilson 1971)

Almost every finite game (all games outside a closed set of measure zero in payoff space) has a finite and *odd* number of Nash equilibria.

This explains a pattern you may have noticed: Battle of the Sexes has 2 pure +1 mixed = 3; coordination games likewise; Matching Pennies has exactly 1.

Exercises

Exercise 5.1: Penalty kicks

A kicker chooses Left or Right; so does the goalkeeper. Scoring probabilities for the kicker: 0.6 if both choose the same side (keeper guesses right), 0.95 if kicker goes Left and keeper Right, 0.9 if kicker Right and keeper Left. Model as a zero-sum game (kicker maximizes scoring probability), find the mixed equilibrium, and compute the equilibrium scoring probability.

Exercise 5.2: Rock–Paper–Scissors

Show that uniform randomization by both players is the unique Nash equilibrium of Rock–Paper–Scissors (win $= 1$, loss $= -1$, tie $= 0$). Then modify the game so a win with Rock pays 2 instead of 1 (for both players' Rock wins) and recompute the equilibrium. Which action's probability *falls*? Explain the intuition.

Exercise 5.3: Volunteer's dilemma

Each of n neighbors can call the police (cost c) or not. If at least one calls, every neighbor gets benefit $v > c$; if nobody calls, all get 0. Find the symmetric mixed equilibrium probability of calling, and show that the probability that *nobody* calls *increases* with n. Discuss the connection to the bystander effect.

Exercise 5.4: Support checking

	L	R
T	$6, 2$	$0, 6$
M	$2, 4$	$4, 2$
B	$3, 3$	$3, 3$

Find *all* Nash equilibria (pure and mixed), checking every possible support profile for player 1.