Chapter 8

Bargaining

How is a surplus divided when two parties must agree to realize it? Bargaining theory has two complementary traditions: Nash's *axiomatic* approach (what properties should a solution have?) and Rubinstein's *strategic* approach (model the back-and-forth explicitly and solve for equilibrium). Their eventual convergence is one of game theory's most satisfying results.

8.1 The axiomatic approach: Nash bargaining

Definition 8.1: Bargaining problem

A (two-player) *bargaining problem* is a pair (S, d) where $S \subset \mathbb{R}^2$ is a compact convex set of feasible utility pairs, and $d = (d_1, d_2) \in S$ is the *disagreement point* (what each gets if talks fail), with some $s \in S$ satisfying $s > d$ componentwise. A *bargaining solution* is a map $f(S, d) \in S$.

Theorem 8.1: Nash bargaining solution (1950)

There is exactly one bargaining solution satisfying:

1. **Pareto efficiency**: no feasible point dominates $f(S, d)$;
2. **Symmetry**: if S is symmetric and $d_1 = d_2$, then $f_1 = f_2$;
3. **Invariance to affine transformations** of either player's utility;
4. **Independence of irrelevant alternatives (IIA)**: if $T \subseteq S$ and $f(S, d) \in T$, then $f(T, d) = f(S, d)$.

It is the point maximizing the *Nash product*:

$$f(S, d) \;=\; \arg\max_{s \in S,\; s \geq d} (s_1 - d_1)(s_2 - d_2).$$

Example 8.1: Splitting a dollar with risk aversion

Two players split \$1; utilities are $u_1(x) = x$ and $u_2(x) = x^\alpha$ with $\alpha \in (0, 1)$ (player 2 is risk-averse), $d = (0, 0)$. Maximizing $x \cdot (1 - x)^\alpha$ gives $x^* = \frac{1}{1+\alpha}$. As α falls (more risk aversion), player 2's share $\frac{\alpha}{1+\alpha}$ *shrinks*: risk aversion is a bargaining weakness. Impatience will play the identical role in the strategic model.

The generalized (asymmetric) Nash solution maximizes $(s_1 - d_1)^{\beta}(s_2 - d_2)^{1-\beta}$, where β captures "bargaining power"—a black box that the strategic approach opens.

8.2 The strategic approach: alternating offers

Definition 8.2: Rubinstein's alternating-offers game (1982)

Players alternate proposing divisions $(x, 1-x)$ of a pie of size 1: player 1 proposes in periods $0, 2, 4, \ldots$, player 2 in periods $1, 3, 5, \ldots$. A proposal accepted in period t gives discounted payoffs $(\delta_1^t x,\ \delta_2^t(1-x))$, where $\delta_i \in (0,1)$ are discount factors. Rejection moves to the next period; perpetual disagreement gives $(0, 0)$.

Theorem 8.2: Rubinstein (1982)

The alternating-offers game has a *unique* subgame perfect equilibrium. Agreement is immediate; player 1 proposes and player 2 accepts

$$x^* = \frac{1-\delta_2}{1-\delta_1\delta_2}, \qquad 1 - x^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}.$$

Proof idea (stationary equilibrium + uniqueness sketch). Look for values (x, y): x is what the current proposer gets when player 1 proposes, y what player 2 gets as proposer. A responder accepts anything at least as good as her continuation value from proposing next period:

$$1 - x = \delta_2\, y, \qquad 1 - y = \delta_1\, x.$$

Solving: $x = \frac{1-\delta_2}{1-\delta_1\delta_2}$, $y = \frac{1-\delta_1}{1-\delta_1\delta_2}$. Uniqueness (Shaked–Sutton argument): let $\bar{x}, \underline{x}$ be the supremum and infimum of player 1's SPE payoffs as proposer. Optimal play bounds $\bar{x} \le 1 - \delta_2(1 - \delta_1\bar{x})$ and $\underline{x} \ge 1 - \delta_2(1 - \delta_1\underline{x})$, which force $\bar{x} = \underline{x} = x^*$. □

Key Idea: What determines bargaining power

In Rubinstein's model, shares are pinned down by *relative patience*: as $\delta_2 \to 0$ (player 2 desperate to settle), player 1 takes almost everything; with equal discounting $\delta_1 = \delta_2 = \delta$, player 1 gets $\frac{1}{1+\delta} > \frac{1}{2}$—a vanishing first-mover advantage as $\delta \to 1$. Outside options, risk of breakdown, and commitment tactics all map into the same calculus. "Bargaining power" is no longer a free parameter but an implication of the environment.

Theorem 8.3: Convergence (Binmore–Rubinstein–Wolinsky 1986)

As the time between offers shrinks to zero, the unique SPE outcome of the alternating-offers game converges to the (generalized) Nash bargaining solution, with disagreement point given by the payoffs during delay/breakdown and bargaining weights determined by relative discount rates.

This "Nash program"—grounding axiomatic solutions in explicit non-cooperative games—is a template used throughout cooperative theory (Chapter 15).

8.3 Bargaining with outside options

Suppose player 2 can leave and take an outside option worth w at any response node.

Proposition 8.4: Outside option principle

If $w \leq \delta_2 y^*$ (the option is worse than 2's equilibrium continuation), the equilibrium is unchanged—the option is an *empty threat*. If $w > \delta_2 y^*$, player 2 receives exactly w: the option binds but confers no extra rent.

The lesson—threats affect outcomes only when credible, and then only at their exact value—recurs in wage negotiation, litigation-vs-settlement, and M&A practice. It also warns against the naïve "split the difference plus my outside option" intuition.

8.4 Applications

Example 8.2: Union–firm wage bargaining

A firm and a union bargain over the wage w; production yields revenue R per period, the union's strike payoff is s per period. With alternating offers and a common discount factor near 1, the outcome approximates the Nash solution with disagreement $(0, s)$: $w^* \approx \frac{R+s}{2}$. The union's strike fund s raises the wage with weight $\frac{1}{2}$ even though no strike ever occurs in equilibrium. *Strength that is never used still shapes the deal.*

Example 8.3: Pre-trial settlement

Litigation will cost each party c and transfer J from defendant to plaintiff with probability p (common knowledge). The settlement surplus relative to trial is $2c$; Nash bargaining predicts settlement at pJ plus/minus the split of costs saved. Settlement fails in practice mainly through *asymmetric information* (each side optimistic about p)—the subject of Chapters 10 and 11, and a leading explanation for costly disputes, strikes, and wars.

Exercises

Exercise 8.1: Nash product computation

S is the triangle with vertices $(0,0)$, $(4,0)$, $(0,2)$; $d = (1, \frac{1}{2})$. Compute the Nash bargaining solution. Verify IIA by re-solving on the smaller set $S' = S \cap \{s_1 \leq 3\}$ and checking the solution is unchanged (or explain if it moved).

Exercise 8.2: Three rounds of alternating offers

Solve by backward induction the three-period alternating-offers game (player 1 proposes at $t = 0$ and $t = 2$, player 2 at $t = 1$; if the $t = 2$ proposal is rejected both get 0), with common discount factor δ. Verify that as the horizon grows, proposer shares approach $\frac{1}{1+\delta}$.

Exercise 8.3: Impatience vs. risk of breakdown

In the alternating-offers game, replace discounting with an exogenous breakdown risk: after each rejection, with probability $1 - \pi$ negotiations collapse and players receive (b_1, b_2). Derive the analogue of the Rubinstein shares and interpret the limit $\pi \to 1$.

Exercise 8.4: Deadline effect

Modify the alternating-offers game so the pie disappears after period T (no discounting). Show the last proposer takes (almost) everything, and explain why real negotiations so often settle "on the courthouse steps."